Sums and Differences of Correlated Random Sets

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Introduction

Given $A \subset \mathbb{Z}$, let

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Theorem

There exists a positive constant c such that for any n large, the proportion of sets $A \subset \{0, \ldots, n\}$ with $|A + A| > |A - A|$ is greater than c. (Martin and O'Bryant 2006)

Such sets are called *More Sums Than Differences (MSTD) sets*, or *sum-dominant sets*.

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Let $\vec{\rho} = (p, \rho_1, \rho_2)$. We call (A, B) a *$\vec{\rho}$-correlated pair.*

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- $\rho_1 = \rho_2, \implies (A, B)$ independent.

Probability function

Let $P(\vec{\rho}, n)$ be the probability that a $\vec{\rho}$-correlated pair $(A, B) \subset \{0, \dots, n\}$ is MSTD, that is

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Note: $P(1/2, 1, 0, n)$, $P(1/2, 0, 1, n)$, and $P(1/2, 1/2, 1/2, n)$ can be thought of as *proportions* of pairs (A, A), (A, A^c), resp. (A, B) which are MSTD, while other values of $P(\vec{\rho}, n)$ must be thought of as *probabilities.*

Main Results on Correlated Pairs

Theorem

For any $\vec{\rho} \in [0,1]^3$*, the limit*

$$\lim_{n\to\infty} P(\vec{\rho}, n) =: P(\vec{\rho})$$

exists. Moreover, as long as $p \notin \{0,1\}$ *and* $(\rho_1, \rho_2) \neq (0,0), (1,1)$*, then* $P(\vec{\rho})$ *is strictly positive.*

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If $\rho_1 = 0$, but $\rho_2 p > 0$, we can use the fringe profile $L = R = \{1, 2, 3, 5, 7, 8\}$, $L' = R' = L^c$. (This means that the left and the right edges of A look like

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and

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For any ρ_1, ρ_2, the function $P(p, \rho_1, \rho_2)$ is a differentiable function of $p \in [0,1]$.

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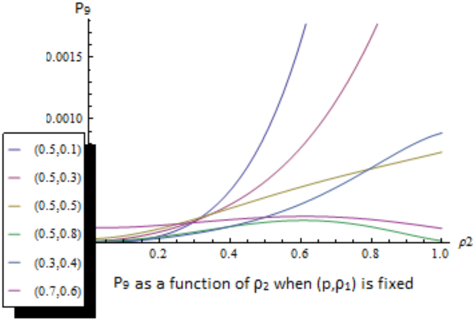
We fix $n = 9$, do an exhaustive search to find all MSTD pairs and calculate P_9.

Fix (p, ρ_1)



P_9 as a function of ρ_2 when (p, ρ_1) is fixed

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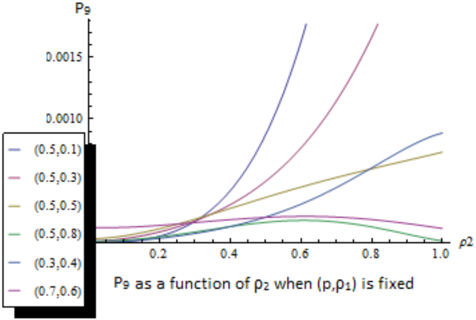
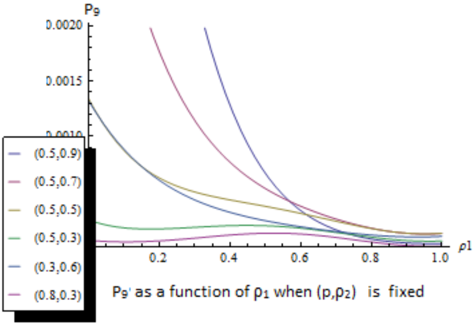


P_9 as a function of ρ_2 when (p,ρ_1) is fixed

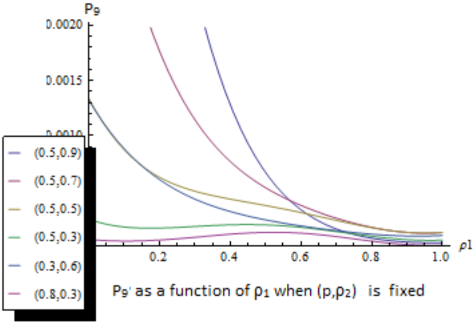
Conjecture 1: For any fixed (p, ρ_1) with ρ_1 not too big ($\rho_1 \leq 0.4$) then P as a function of ρ_2 is strictly increasing in $[0, 1]$ and reaches its maximum at $\rho_2 = 1$.

Fix (p, ρ_2)



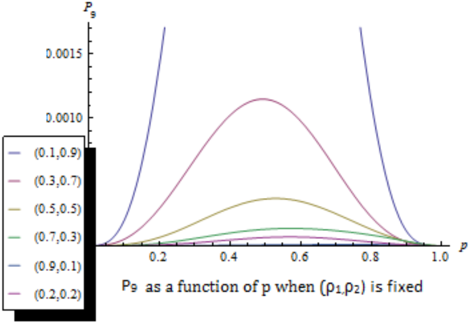
P_9' as a function of ρ_1 when (p, ρ_2) is fixed

Fix (p, ρ_2)



Conjecture 2: For any fixed (p, ρ_2) with ρ_2 not too small ($\rho_2 \geq .5$) then P as a function of ρ_1 is strictly decreasing in $[0, 1]$ and reaches its maximum at $\rho_1 = 0$.

Fix (ρ_1, ρ_2)



P_9 as a function of p when (ρ_1, ρ_2) is fixed

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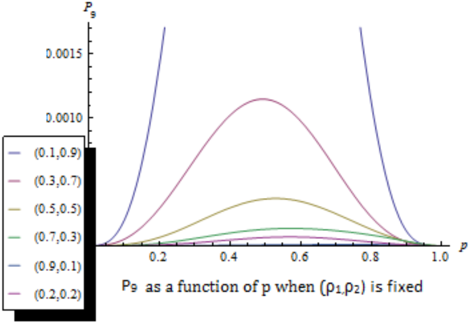

P_9 as a function of p when (ρ_1, ρ_2) is fixed

Conjecture 3: For any fixed (ρ_1, ρ_2), P as a function of p in $(0, 1)$ has a maximum at $1/2$.

A and *A* complement

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Conjecture 4: The maximum of the function $P(p, \rho_1, \rho_2)$ in $[0, 1]^3$ occurs at $P(1/2, 0, 1) \approx 0.03$.

Some notation

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- $X \sim f(N)$ if for any $\epsilon_1, \epsilon_2 > 0$ there exists $N_{\epsilon_1,\epsilon_2} > 0$ such that for all $N > N_{\epsilon_1,\epsilon_2}$

$$P\Big(X \notin [(1-\epsilon_1)f(N), (1+\epsilon_1)f(N)]\Big) < \epsilon_2$$

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Hegarty-Miller investigated this for $(\rho_1, \rho_2) = (1, 0)$ and $p = p(n) = o(1), n^{-1} = o(p(n))$. The first condition indicates p decays with n while the second one guarantees the expected size of A grow with n.

Theorem (Hegarty-Miller)

Given a function $p : \mathbb{N} \to (0, 1)$ such that $p(N) = o(1)$ and $N^{-1} = o(p(N))$. As $N \to \infty$, the probability A as a subset of $[1, \ldots, N]$ is *MSTD* tends to 0. Let $\mathcal{S} = |A + A|, \mathcal{D} = |A - A|$ and $\mathcal{S}^C, \mathcal{D}^C$ denote their complements.

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(ii) $p = c.N^{-1/2}$ for $c \in (0, \infty)$. Let $g(x) = 2(e^{-x} - (1-x))/x$:

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(iii) $N^{-1/2} = o(p)$: $\mathcal{S}^C \sim 2.\mathcal{D}^C \sim \frac{4}{p^2}$

Our result

We prove a similar result:

Let $\hat{p} = p^2(2\rho_1 - \rho_1^2) + 2p(1-p)\rho_2$ be depend on N. Then

(i) $\hat{p} = o(N^{-1})$: Then $\mathcal{D} \sim 2\mathcal{S} \sim N^2.\hat{p}$

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(iii) $N^{-1} = o(\hat{p})$: $\mathbb{E}(\mathcal{S}^C) = \mathbb{E}(2.\mathcal{D}^C) = 4/\hat{p}$

Notes in Our result

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The interesting case is A and A^C: $\hat{p} = 2p(1-p) = \Theta(1/N)$. If we let $p = o(1)$, $p = \Theta(1/N)$ which implies the expected number of elements of A is $pN =$ constant.

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Examples of minimal size MSTD pair:

$$A = \{1, 2, 5, 7\}, \quad B = \{1, 3, 6, 7\}$$

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Corollary: There is no MSTD pair of size $(1, k)$ and $(2, k)$ for $k > 0$.

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We use some tedious checking to eliminate the case $(3, 3)$ and $(3, 4)$.

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- The function $P(\vec{\rho})$ is continuous and differentiable.
- We show that $P(\vec{\rho})$ approaches zero and characterize the phase transition when we let $\vec{\rho}$ decay with n.

Summary of Results

- We prove for each $\vec{\rho} = (p, \rho_1, \rho_2)$ the limiting probability $P(\vec{\rho})$ of picking an MSTD $\vec{\rho}$-correlated pair exists and positive (except in some extreme cases).
- The function $P(\vec{\rho})$ is continuous and differentiable.
- We show that $P(\vec{\rho})$ approaches zero and characterize the phase transition when we let $\vec{\rho}$ decay with n.
- We find the minimal size of an MSTD pair (A, B).

Future Research

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- Prove the uniqueness of the MSTD pairs of size $(4, 4)$ and $(3, 5)$, up to translation/dilation.

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