Benford Behavior of Dependent Random Variables

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Summary

- Benford's Law
- Stick Decomposition
- Conjectures and Other Dependent Systems

Consider the dataset of the populations of Spanish cities.
How often do you expect a leading digit of 1 to occur?

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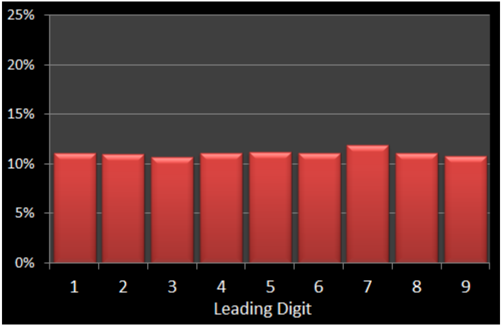
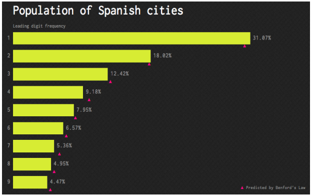


First Digit Bias

Population of Spanish Cities



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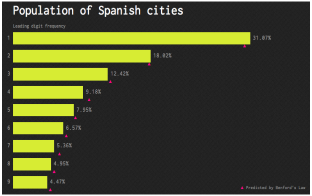
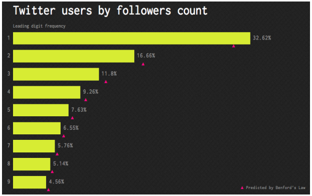


Twitter Followers per User



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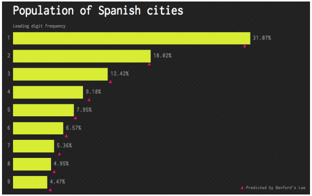

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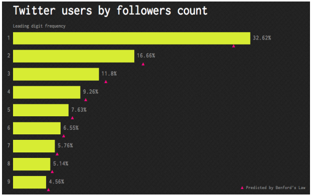
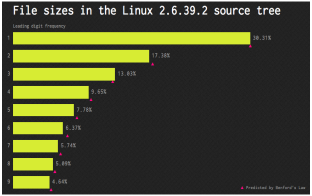


File Sizes in Linux Source Tree



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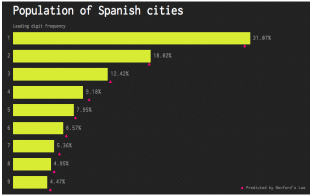

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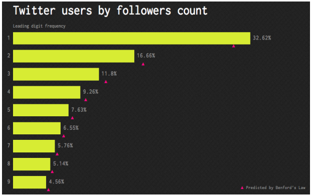

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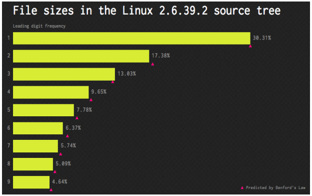
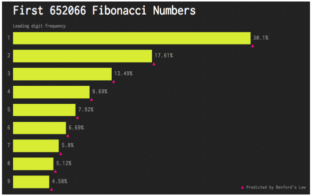


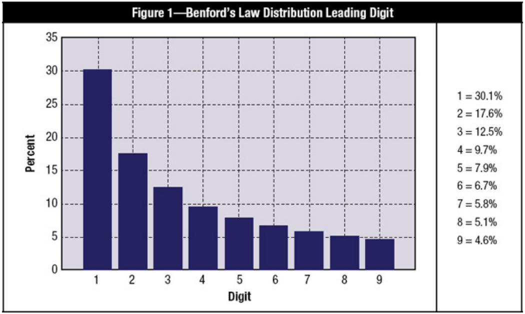
Fibonacci numbers



Benford's Law

Definition

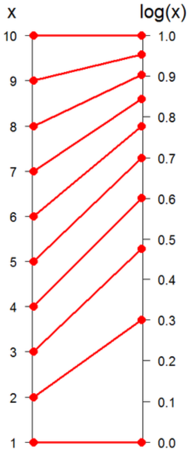
A dataset is said to follow **Benford's Law** (base b) if the probability of observing a first digit of d is $\log_b \frac{1+d}{d}$.



Logarithms and Benford's Law

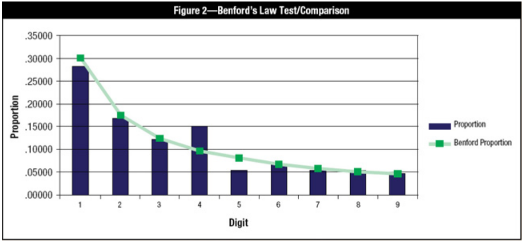
P(leading digit d) $=$ $\log_{10}(d+1) - \log_{10}(d)$

Benford's law $\leftrightarrow$ mantissa of logarithms of data are uniformly distributed



Applications of Benford's Law

- Fraud detection
- Data integrity
- Analyzing round-off errors



Previous Work

- Arithmetic operations on random variables.
- Reliance on independence of random variables.
- Becker, Greaves-Tunnell, Miller, Ronan, Strauch: techniques to deal with dependencies.
- Lemons: process of fragmenting a conserved quantity.

Fixed Proportion Decomposition

Fixed Proportion Decomposition Process

Decomposition Process

1. Consider a stick of length $\mathcal{L}$.

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3. Break the stick into two pieces: lengths $p\mathcal{L}$ and $(1 - p)\mathcal{L}$.

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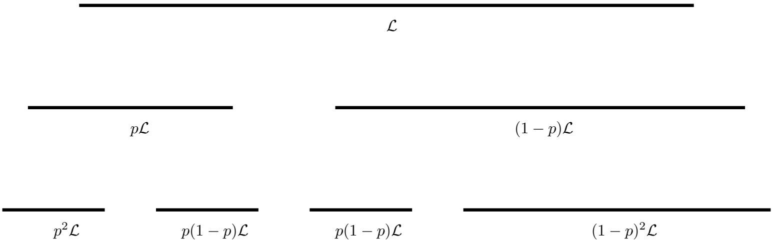
4. Repeat N times (using the same proportion).

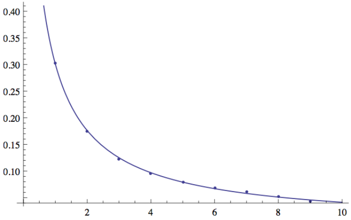
Fixed Proportion Decomposition Process



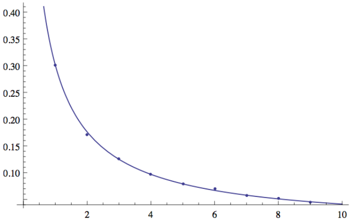
Fixed Proportion Conjecture

Joy Jing's Conjecture

The above decomposition process results in stick lengths that obey Benford's Law as $N \to \infty$ for any $p \in (0, 1)$, $p \neq \frac{1}{2}$.

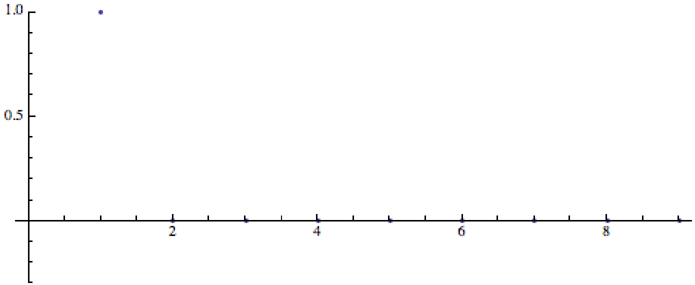


(B) $p = 0.51$ and $N = 10000$.



(B) $p = 0.99$ and $N = 50000$. Benford distribution overlaid.

Counterexample: $p = \frac{1}{11},\ 1 - p = \frac{10}{11}$.



Benford Analysis

After Nth interation,

- 2^N sticks
- $N+1$ distinct lengths.

Distinct lengths are given by

$$x_{j+1} \;=\; \left(\frac{1-p}{p}\right) x_j,\;\; x_0 = p^N.$$

Frequency of $x_j = \binom{N}{j}$

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Frequency of $x_j = \binom{N}{j}$

Let $\frac{1-p}{p} = 10^y$.

$\frac{1-p}{p} = 10^y$, **y** $\in \mathbb{Q}$

Theorem

Let $\frac{1-p}{p} = 10^y$. If $y \in \mathbb{Q}$, the described decomposition process results in stick lengths that do not obey Benford's Law.

$\frac{1-p}{p} = 10^y$, y $\in \mathbb{Q}$

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Let $y = \frac{r}{q}$.

Leading digit of x_j repeats every q indices. Thus,

$$\sum_k P(x_{j+kq}) = \sum_k \binom{N}{j+kq}.$$

Series Multisection

Multisection Formula

If $f(x) = \sum_{n=-\infty}^{\infty} a_n x^n$,

$$\sum_{k=-\infty}^{\infty} a_{kq+j} x^{kq+j} = \frac{1}{q} \sum_{p=0}^{q-1} \omega^{-jp} f(\omega^p x)$$

where ω is the primitive qth root of unity $e^{2\pi i/q}$.

Multisection of Binomial Coefficients

$$\sum_k \binom{N}{j+kq} = \frac{2^N}{q} \sum_{s=0}^{q-1} \left(\cos \frac{\pi s}{q}\right)^N \cos \frac{\pi(N-2j)s}{q}.$$

$\frac{1-p}{p} = 10^y,\ y \in \mathbb{Q}$

$$\sum_k P(x_{j+kq}) = \frac{1}{q}\left(1 + \mathscr{E}\left[(q-1)\left(\cos\frac{\pi}{q}\right)^N\right]\right)$$

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$$\sum_k P(x_{j+kq}) = \frac{1}{q}\left(1 + \mathscr{E}\left[(q-1)\left(\cos\frac{\pi}{q}\right)^N\right]\right)$$

Digit frequencies are multiples of $\frac{1}{q}$.

Benford frequencies are irrational, so *not* perfect Benford.

$\frac{1-p}{p} = 10^y$, y $\notin \mathbb{Q}$: Outline

Theorem

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$\{x_j\} \sim Bin(N, \frac{1}{2})$

mean: $\frac{N}{2}$

standard deviation: $\frac{\sqrt{N}}{2}$

Outline of proof strategy:

1. Truncation
2. Break into intervals
   - Roughly equal probability
   - Equidistribution

$\frac{1-p}{p} = 10^y$, $y \notin \mathbb{Q}$: Truncation

For $\epsilon > 0$, Chebyshev's Inequality gives

$$\begin{aligned} P\left(\left|x - \frac{N}{2}\right| \geq N^{\frac{1}{2}+\epsilon}\right) &= P\left(\left|x - \frac{N}{2}\right| \geq N^{\epsilon} N^{\frac{1}{2}}\right) \\ &\leq \frac{1}{N^{2\epsilon}}. \end{aligned}$$

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So we can limit our analysis to

- N^{ϵ} standard deviations
- Right half of binomial

$\frac{1-p}{p} = 10^y$, y $\notin \mathbb{Q}$: Intervals and Roughly Equal Probability

$\mathcal{I}_\ell = \{x_\ell, x_\ell + 1, \dots, x_\ell + N^\delta - 1\}$.

Let $x_0 = N/2$. It follows that $x_\ell = N/2 + \ell N^\delta$.

$$\left| \binom{N}{x_\ell} - \binom{N}{x_{\ell+1}} \right| \leq \binom{N}{x_\ell} N^{-\frac{1}{2}+\delta+\epsilon},$$

when $\delta < 1/2 - \epsilon$ and $\ell \leq N^{1/2-\delta+\epsilon}$.

$\frac{1-p}{p} = 10^y$, y $\notin \mathbb{Q}$: Equidistribution

Definition

$\{x_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if for any $[a, b] \subset [0, 1]$, $P\left(x_n \mod 1 \in [a, b]\right) \to b - a$:

$$\lim_{N\to\infty} \frac{\#\{n \leq N : x_n \mod 1 \in [a, b]\}}{N} = b - a.$$

Recall: Leading digits of stick lengths are Benford if their logarithms are equidistributed modulo 1.

$\frac{1-p}{p} = 10^y$, y $\notin \mathbb{Q}$: Equidistribution

Consider an interval I_ℓ where

$$I_\ell = \{x_\ell + i : i \in \{0, 1, \dots, N^\delta - 1\}\}$$

$$J_\ell \subset \{0, 1, \dots, N^\delta - 1\} = \{i : \log(x_\ell + i) \bmod 1 \in [a, b]\}.$$

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If the irrationality exponent κ of y is finite,

$$|J_\ell| \;=\; (b-a)N^\delta + O(N^{\delta(1-\frac{1}{\kappa}+\epsilon)})$$

$\frac{1-p}{p} = 10^y$, y $\notin \mathbb{Q}$: Equidistribution

Using

- equidistribution within intervals
- roughly equal probability

we have

$$\sum_{\ell} \sum_{i \in J_\ell} f(x_\ell + i) \quad = \quad (b-a) + O(N^{\delta(-\frac{1}{\kappa}+\epsilon)} + N^{-\frac{1}{2}+\delta+\epsilon}).$$

where the irrationality exponent κ of y is finite.

Additive Decomposition

Additive Decomposition Process

Decomposition Process

1. Consider a stick of length $\mathcal{L}$.
2. Break the stick into two pieces, both of integer length.
3. Freeze a piece if its length exists in the specified stopping sequence.
4. Repeat decomposition with pieces that have not been frozen.

Additive Decomposition Processes: Conjectures

Benford

- Stop at evens (proved)
- Stop at primes

Non-Benford

- Stop at squares
- Stop at powers of two
- Stop at powers of three
- Stop at Fibonnaci numbers

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Continuous Model Stick Labeling

Iteration 0: $\mathcal{L}$

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Iteration 2: $p_2p_1\mathcal{L}$ $x_2 = (1 - p_2)p_1\mathcal{L}$

Continuous Model Stick Labeling

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Iteration N: $p_N p_{N-1} \cdots p_1\mathcal{L}$ $\qquad x_N = (1 - p_N)p_{N-1} \cdots p_1\mathcal{L}$

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Continuous Model

Approach:

- Draw cut proportions from uniform distributon on $(0, 1)$.
- Label sticks according to number of proportions in their product.
- Do not consider first $\log N$ sticks as well as pairs of sticks x_i, x_j where i and j differ by less than $\log N$.
- Show $\mathbb{E}[P_N(s)] \to \log s$ and $\mathrm{Var}[P_N(s)] \to 0$.

Other Work and Conclusions

Determinant Expansions

Theorem

Let A be an $n \times n$ matrix with i.i.d. entries $a_{ij} \sim X$ with density f. The $n!$ terms in the determinant expansion of A are Benford if

$$\lim_{n\to\infty} \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \prod_{m=1}^{n} M_f\left(1 - \frac{2\pi i l}{\log 10}\right) = 0$$

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Important Technique: Quantify dependencies among terms.

Conclusions

- Dependent vs Independent Random Variables
- Stick Decomposition
  - Fixed Proportion Decomposition $\frac{1-p}{p} = 10^y$

    - $y \in \mathbb{Q}$, not Benford

    - $y \notin \mathbb{Q}$, Benford (finite vs infinite κ)
  - Additive Stick Decomposition

    - Conjectures

    - Continuous Model
- Determinant Expansion

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Irrationality Exponent

Let $y \in \mathbb{R}$. Denote by $\mathcal{A}$ the set of positive numbers κ for which

$$0 \leq \left| y - \frac{p}{q} \right| \leq \frac{1}{q^n}$$

has at most finitely many solutions for $p, q \in \mathbb{Z}$.
The irrationality measure of y, denoted $\kappa(y)$, is $\inf_{\kappa \in \mathcal{A}} \kappa$.

If $\mathcal{A}$ is empty, $\kappa(y) = \infty$
For nonempty $\mathcal{A}$,

$$\kappa(y) = \begin{cases} 1 \text{ if } y \text{ is rational} \\ 2 \text{ if } y \text{ is algebraic of degree } > 1 \\ \geq 2 \text{ if } y \text{ is transcendental} \end{cases}$$

Proof of Multisection Formula I

As $f(x) = \sum_{n=0}^{\infty} a_n x^n$,

$$\sum_{j=0}^{k-1} \omega^{-jm} f(\omega^j x) = \sum_{j=0}^{k-1} \omega^{-jm} \sum_{n=0}^{\infty} a_n (\omega^j x)^n$$

$$= \sum_{n=0}^{\infty} a_n \left(\sum_{j=0}^{k-1} \omega^(n-m)j \right) x^n$$

Proof of Multisection Formula II

If $n - m = kl$ for some $l \in \mathbb{Z}$, then using the fact that $\omega^k = 1$ gives

$$\omega^{(n-m)j} = \omega^{klj} = 1$$

which gives $\sum_{j=0}^{k-1} \omega^{(n-m)j} = k$ If $n - m \neq lk$,

$$\sum_{j=0}^{k-1} \omega^{(n-m)j} = \frac{1 - \omega^{(n-m)k}}{1 - \omega^{n-m}} = 0$$

κ Infinite

Let $\alpha \notin \mathbb{Q}$. It is well known that $n\alpha$ mod 1 is equidistributed. For all $[a, b] \subset [0, 1]$, given $\epsilon > 0$, there exists $M(\epsilon, a, b, \alpha)$ such that

$$\#\{n \leq N : n\alpha \bmod 1 \in [a, b]\} \quad = \quad (b - a)N + \mathscr{E}(\epsilon N)\}$$

for all $N \geq M(\epsilon, a, b, \alpha)$.
Now let N be sufficiently large so that N^δ is greater than $M(\epsilon, a, b, \alpha)$.

Quantifying Dependencies

Given a fixed $X_{i,n}$, the number of terms that share k elements is

$$\binom{n}{k}\sum_{\alpha=0}^{n-k}\binom{n-k}{\alpha}(n-k-\alpha)!(-1)^{\alpha}$$

Let $K_{i,j}$ be the number of matrix entries shared by $X_{i,n}$ and $X_{j,n}$. Fixing $X_{i,n}$,

$$\begin{aligned} P(K_{i,j}=k) &= \frac{1}{k!}\sum_{\alpha=0}^{n-k}\frac{(-1)^{\alpha}}{\alpha!} \\ &= \frac{1}{ek!}+O\left(\binom{n}{k}\frac{1}{(n-k)n!}\right) \end{aligned}$$

Quantifying Dependencies

$$\begin{aligned}\mathbb{E}[K_{i,j}] &= \sum_{k=0}^{n-2} \frac{k}{ek!} + \sum_{k=0}^{n-2} O\left(\binom{n}{k}\frac{k}{(n-k)n!}\right) \\ &\to 1.\end{aligned}$$

$$\begin{aligned}\mathrm{Var}\left(K_{i,j}\right) &= \sum_{k=0}^{n-2} \frac{k^2}{ek!} + \sum_{k=0}^{n-2} O\left(\binom{n}{k}\frac{k^2}{n!(n-k)}\right) - 1 \\ &\to 1\end{aligned}$$

Using Chebyshev's Inequality,

$$P(|K_{i,j} - 1| \geq \gamma) \leq 1/\gamma^2$$

References

BGMRS T. Becker, A. Greaves-Tunnell, S. Miller, R. Ronan, and F. Strauch, *Benford's Law and Continuous Dependent Random Variables*. http://arxiv.org/abs/1111.0568.

BLD *Benford's Law Distribution Leading Digit*. Digital image. ISACA. N.p., n.d. Web. 15 June 2013. http://www.isaca.org/Journal/Past-Issues/2011/Volume-3/PublishingImages/11v3-understanding-and.jpg.

C Chen, Hongwei. *On the Summation of First Subseries in Closed Form*. International Journal of Mathematical Education in Science and Technology 41:4, 538-547.

DFD *Distribution of First Digit.* Digital image. Benford's Law. DataGenetics, n.d. Web. 15 July 2013. http://www.datagenetics.com/blog/march52012/.

JJ Jing, Joy. *Benford's Law and Stick Decomposition.* Undergraduate Mathematics Thesis, Williams College. Advisor: Steven Miller.

JKKKM D. Jang, J. U. Kang, A. Kruckman, J. Kudo and S. J. Miller, *Chains of distributions, hierarchical Bayesian models and Benford's Law*, to appear in the Journal of Algebra, Number Theory: Advances and Applications.

KM Kontorovich, Alex V. and Steven J. Miller. *Benford's Law, Values of L-functions and the* $3x + 1$ *Problem.* http://arxiv.org/pdf/math/0412003v2.pdf

MT Miller, Steven J. and Ramin Takloo-Bighash. "Needed Gaussian Integral." *An Invitation to Modern Number Theory.* Princeton: Princeton UP, 2006. 222. Print. http://arxiv.org/pdf/1111.0568v1.pdf

S Singleton, Tommie W., Ph.D. *Benford's Law Test/Comparison.* Digital image. ISACA. N.p., 2011. Web. 16 July 2013. http://www.isaca.org/Journal/Past-Issues/2011/Volume-3/Pages/Understanding-and-Applying-Benfords-Law.aspx.