Sets Characterized by Missing Sums and Differences in $\mathbb{Z}^d$

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Young Mathematicians Conference, 2013
Ohio State University

Introduction

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Typical pair of elements in $a_1, a_2 \in A$ has two differences, but only one sum. Expect sets A with $|A + A| > |A - A|$ to be rare.

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Monte Carlo simulations suggest $\rho \approx 4.5 \cdot 10^{-4}$.

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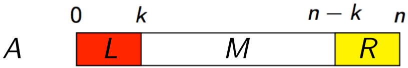

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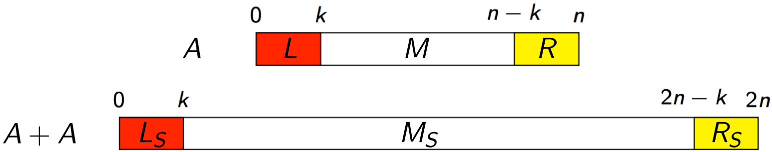

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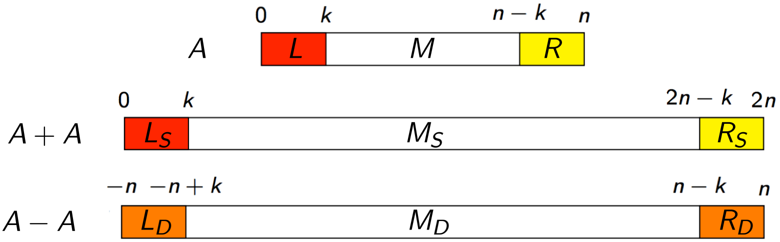

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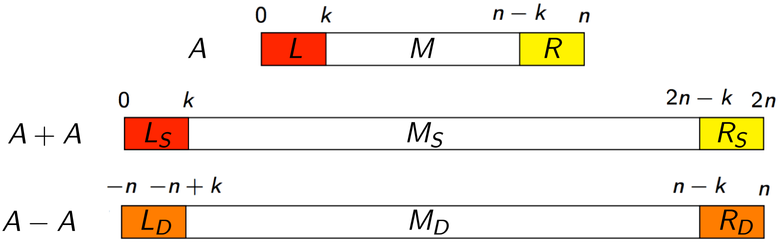

To show that ρ_n is positive, carefully fix elements in L and R so that

$$|(A+A)\cap(L_S\cup R_S)| > |(A-A)\cap(L_D\cap R_D)|.$$

With positive probability independent of n, $M_S \subset A+A$ (and therefore A is MSTD) .

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Theorem [Hegarty, 2007]

For any $s, d \in \mathbb{N}_0$, there exists a constant $c \in (0, 1)$ that, for n sufficiently large depending on (s, d), at least $c \cdot 2^{n+1}$ subsets A of $[0, n]$ satisfy $|A + A| = (2n + 1) - s$ and $|A - A| = (2n + 1) - 2d$.

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In particular, Hegarty explicitly constructs fringe sets L and R such that $(A + A) \cap (L_S \cup R_S)$ is missing s sums and $(A - A) \cap (L_D \cup R_D)$ is missing $2d$ differences.

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With positive probability, $M_S \subset A + A$ and $M_D \subset A - A$ (and therefore $A + A$ and $A - A$ are missing no other sums and differences).

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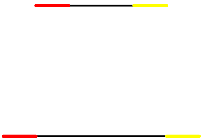
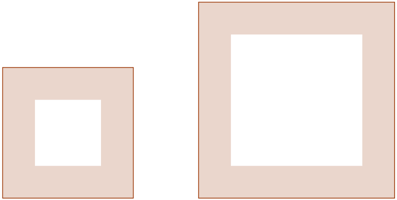
Question

What is the fringe structure?





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Let $D \in \mathbb{N}$. For any $(s, d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, there exists a constant $c_{s,d} \in (0, 1)$ such that, for n sufficiently large depending on (s, d), at least $c_{s,d} \cdot 2^{(n+1)^D}$ subsets of $[0, n]^d$ satisfy $|A + A| = (2n + 1)^D - s$ and $|A - A| = (2n + 1)^D - 2d$.

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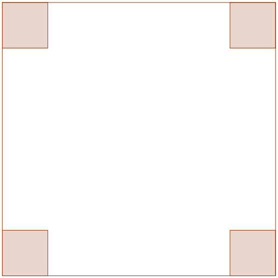
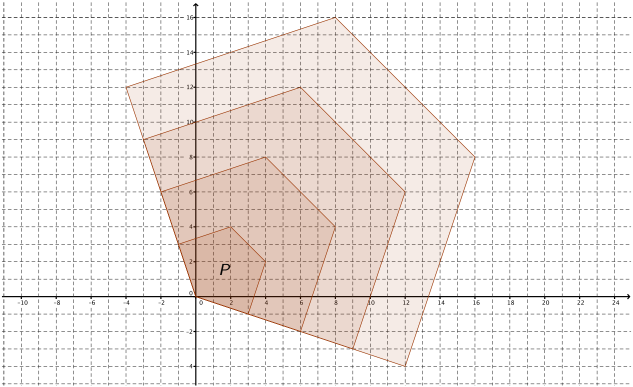
In particular, it suffices to fix a fringe about the corners:



Different Shapes

We can generalize further. Higher dimensions allow for different geometries of our set.

Let P be a convex polytope in $\mathbb{R}^d$ with vertices in $\mathbb{Z}^d$. We consider sets $A \subset L(nP) = nP \cap \mathbb{Z}^d$.



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Let P be a convex polytope in $\mathbb{R}^D$ with vertices in $\mathbb{Z}^D$, and let $s, d \in \mathbb{N}_0$ be given. Then there exists a constant $c > 0$ such that for n large, a uniformly randomly chosen subset of $L(nP)$ has s missing sums and at least $2d$ missing differences with probability at least c.

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Remark

In particular, this theorem implies the existence of a positive percentage of "more missing differences than missing sums" sets, which are in some sense an analogue of MSTD sets.

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- Show that there is a positive probability that every sum is present in the region outside the controlled fringes of the sumset.
- Conclude that we have exactly s sums missing and at least $2d$ differences missing for a positive percentage of sets.

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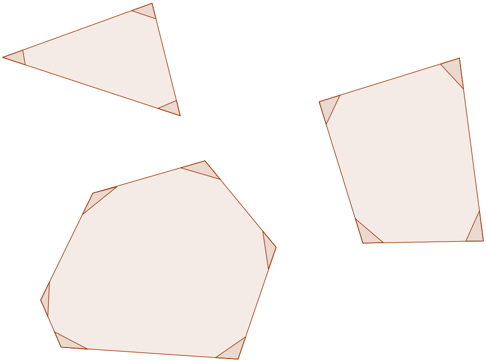
Like in the cube, we fix a fringe of constant size about the corners.



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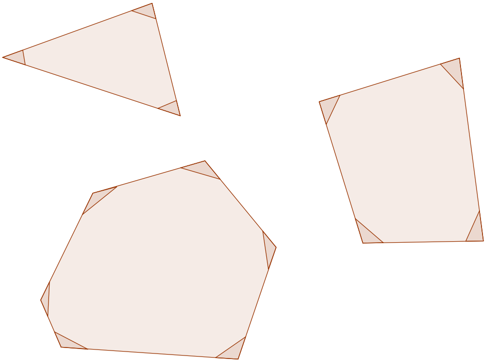
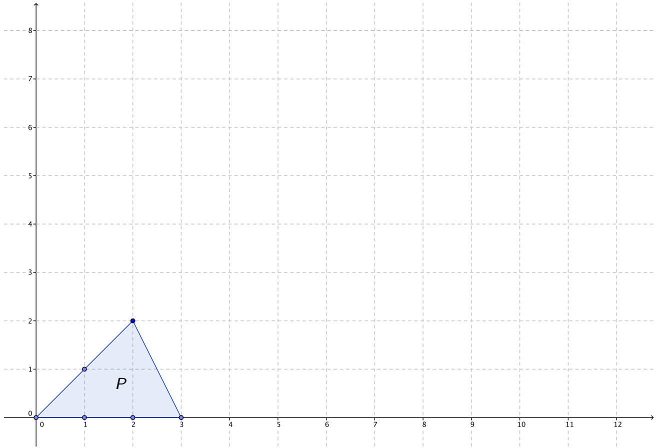


Missing sums and missing differences are controlled on the edges within the corner fringe.

Lattice Points on the Edges



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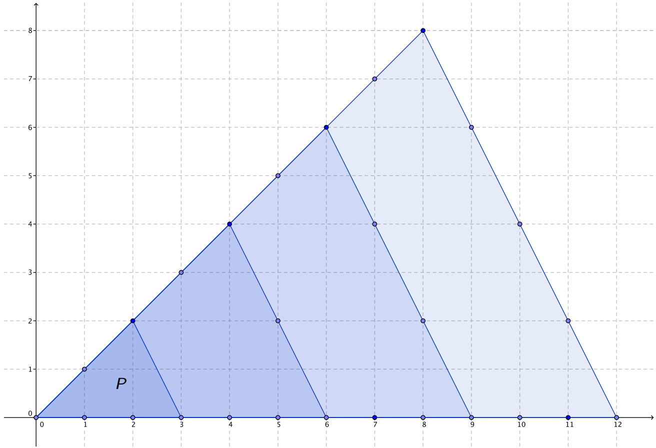

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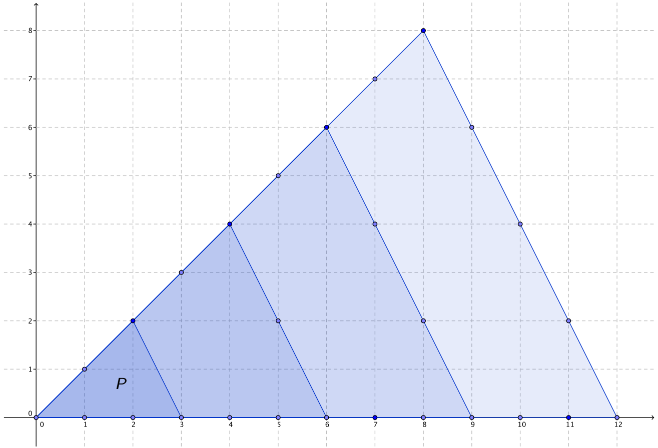
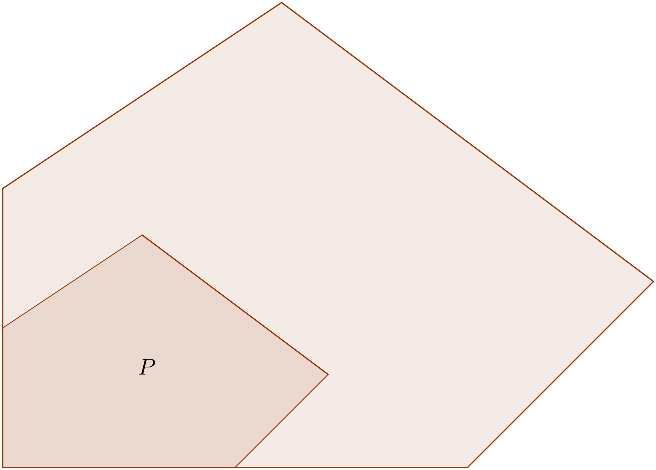


Since vertices are integer lattice points, note that:

- nP has at least $n+1$ lattice points on each edge
- Each edge forms its own 1D lattice structure

Controlling Missing Sums



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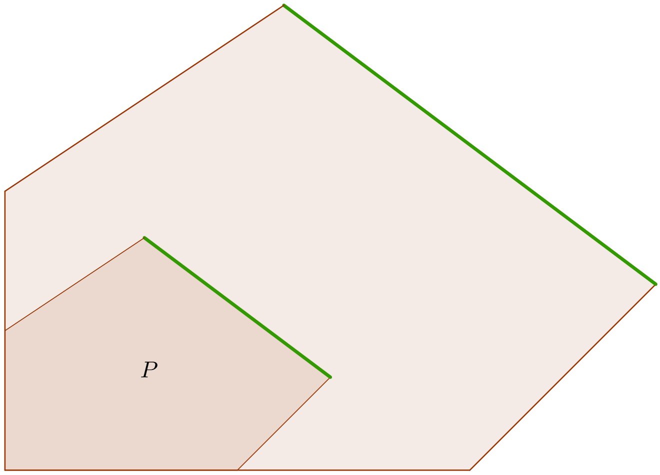

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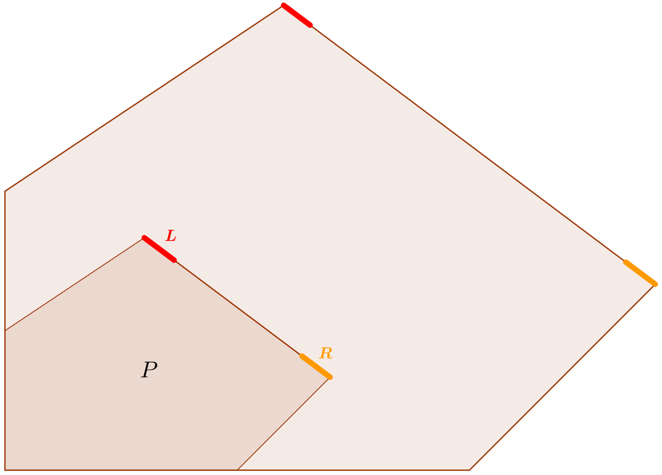

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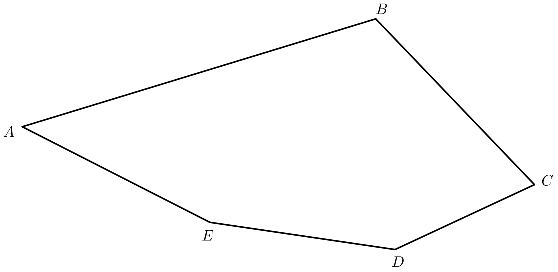

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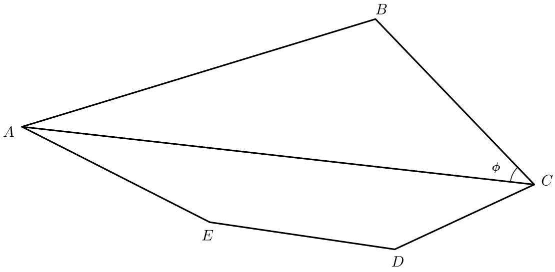

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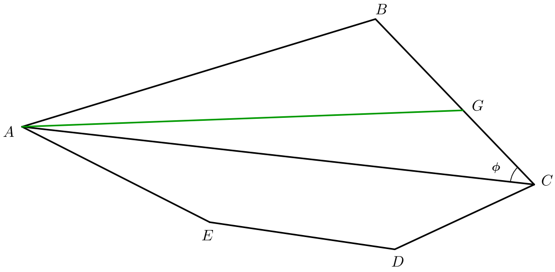

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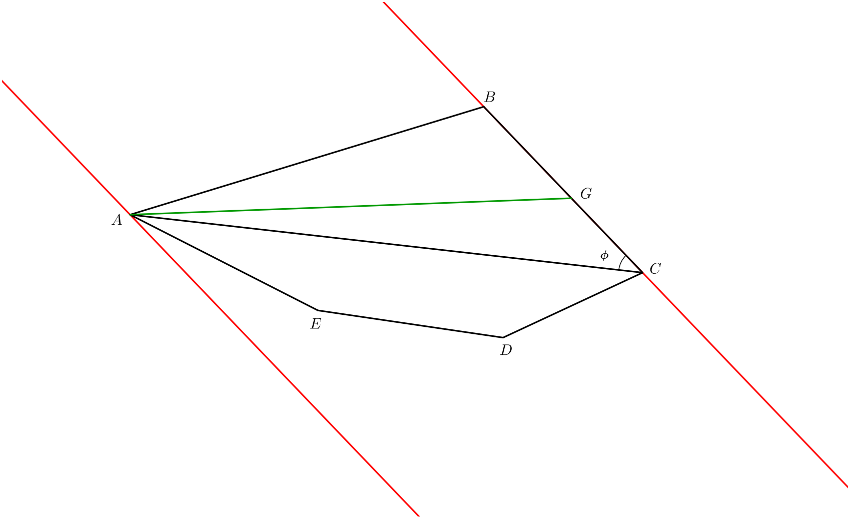

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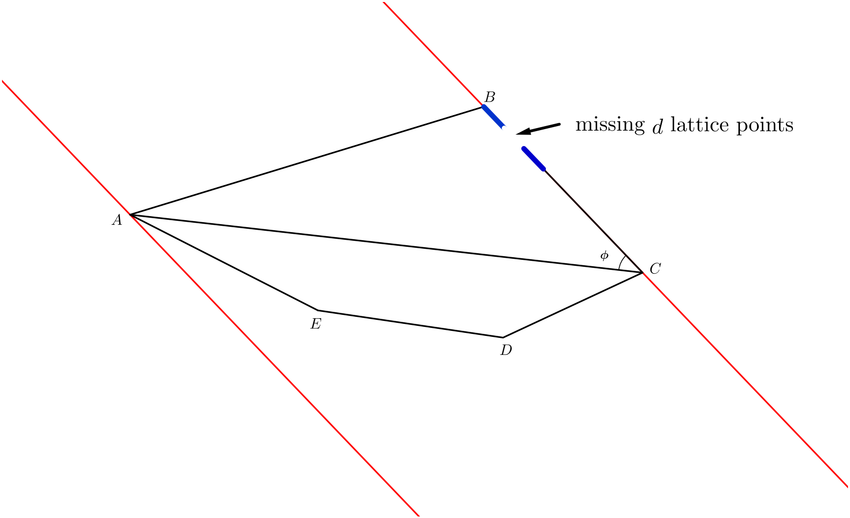
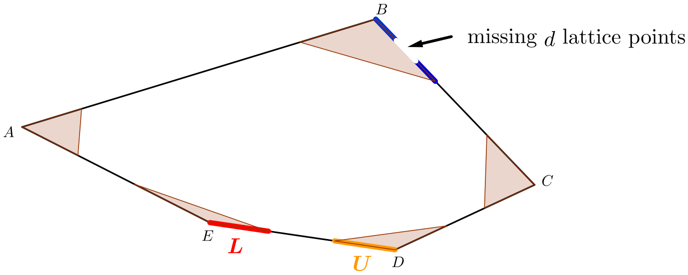


Putting It All Together



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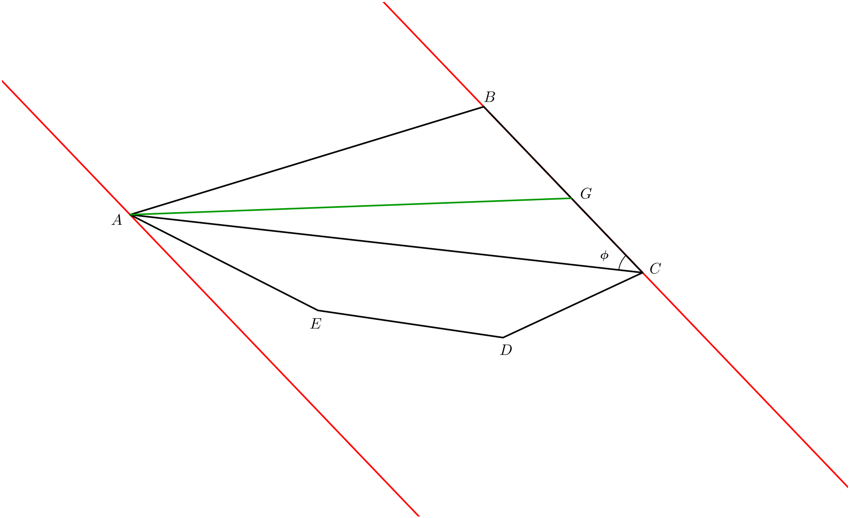

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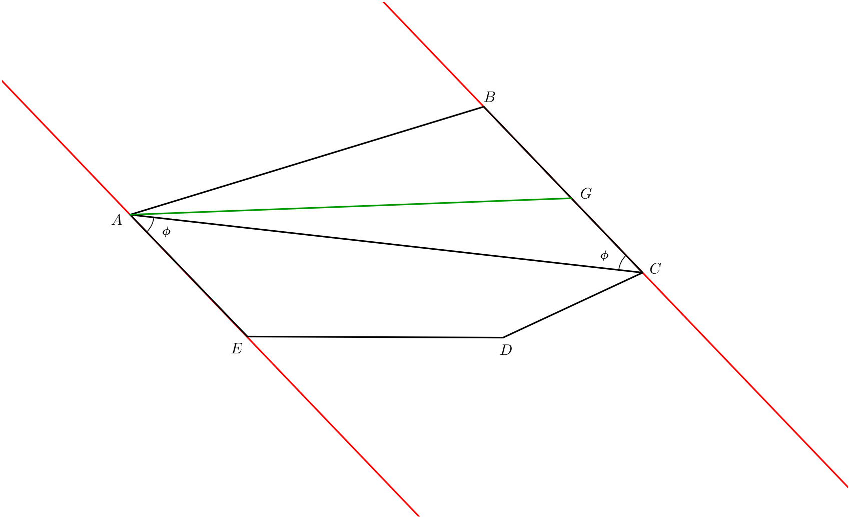

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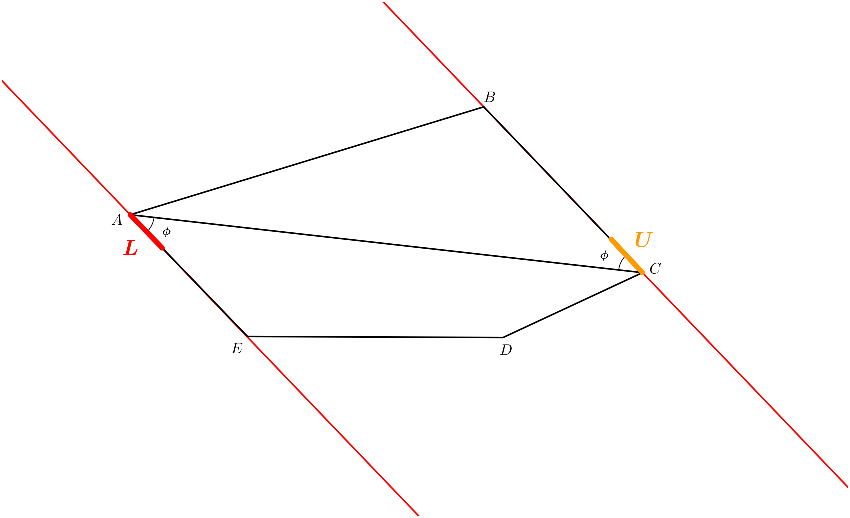

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Let A be a uniformly randomly chosen subset of $L(P)$ with our fixed fringe. Let $\mathbf{k}$ be given in $L((P+P) \setminus F_s)$. Then

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□

Probability that All Middle Sums are Present

We wish to find a constant $c > 0$ so that

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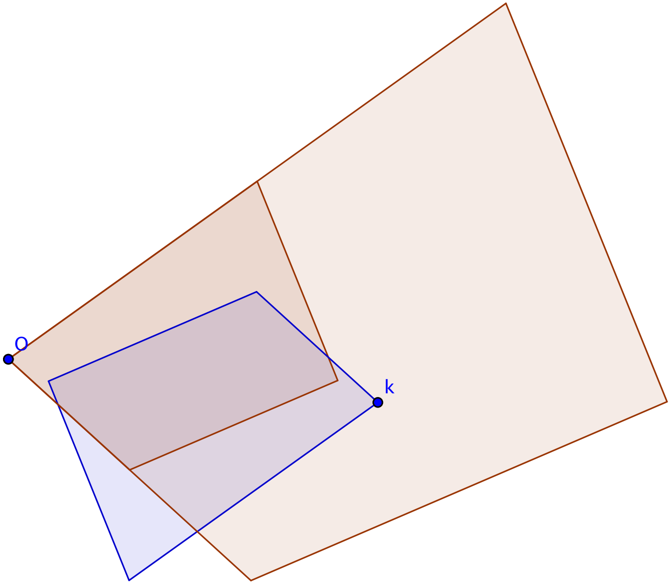
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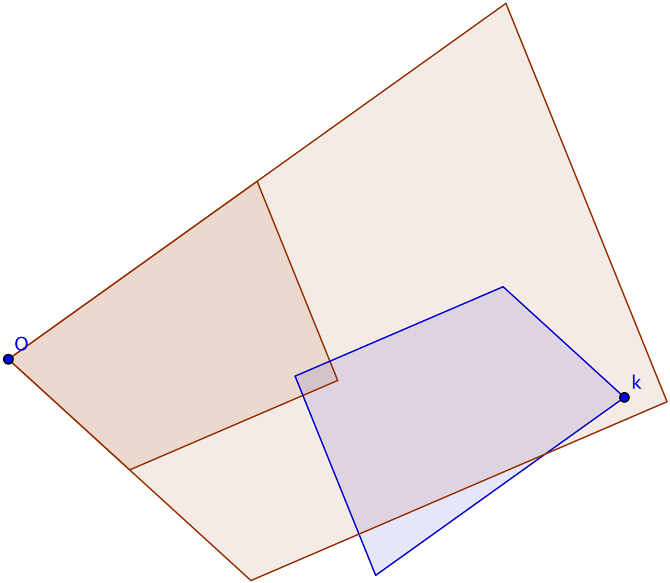
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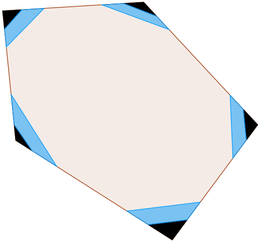
- A middle region where $P \cap (\mathbf{k} - P)$ is so large that the probability of any given sum to be missing is less than $\frac{\epsilon}{|L(P+P)|}$.
- An intermediate region, close enough to the vertices that $P \cap (\mathbf{k} - P)$ is simple enough that its area can be estimated.



Thus, we can fix a large enough fringe so that

$$\sum_{\mathbf{k} \in (L(P)+L(P)) \setminus F_s} \mathbb{P}[\mathbf{k} \notin P + P] < 1 - c,$$

so for n large, all middle sums are present with probability at least c.

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Since our fixed fringe guarantees s missing fringe sums and at least $2d$ missing differences, we have found a positive proportion of subsets with exactly s missing sums and at least $2d$ missing differences overall.

Thanks

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