Newman's Conjecture in Various Settings

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- It is an "almost counter-conjecture" to the Riemann hypothesis!
- We'll look at what happens when we study Newman's conjecture in the function fields setting.

The Riemann zeta function is initially defined, for $\mathrm{Re}(s) > 1$, by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \left(= \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \right).$$

Riemann Hypothesis (1859)

If $\zeta(s) = 0$, then either s is a “trivial zero” or $\mathrm{Re}(s) = \frac{1}{2}$.

Define a new function $\Xi(x)$ for $x \in \mathbb{C}$ as follows:

- Let $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ ("completed zeta function").
- Let $\Xi(x) = \xi\left(\frac{1}{2} + ix\right)$

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Facts:

- If $x \in \mathbb{R}$, then $\Xi(x) \in \mathbb{R}$.
- RH is equivalent to: all the zeros of $\Xi(x)$ are real.

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$\Xi(x)$

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$$\Xi(x) \xrightarrow{\quad 1 \quad} \Phi(u) \xrightarrow{\quad 2 \quad} e^{tu^2}\Phi(u) \xrightarrow{\quad 3 \quad} \Xi_t(x)$$

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- Step 3: Fourier inversion

$$\Xi_t(x) = \int_0^\infty e^{tu^2}\Phi(u)\cos ux\,du.$$

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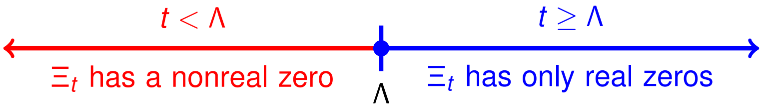
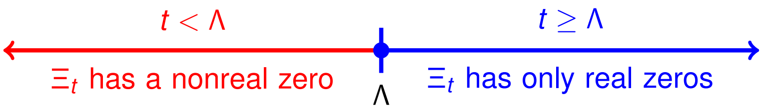


Relationship of Λ to RH



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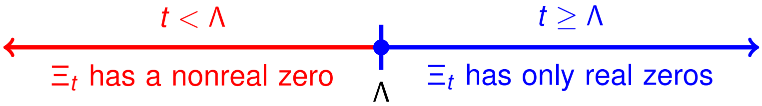

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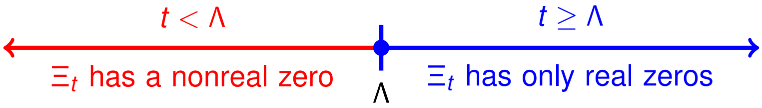

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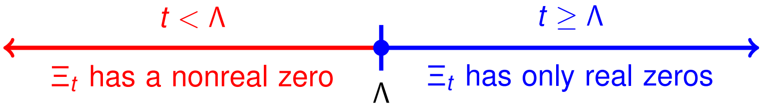

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Conjecture (Newman)

$\Lambda \geq 0$

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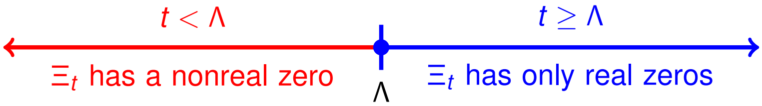

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$\Lambda \geq 0$

Newman: “The new conjecture is a quantitative version of the dictum that the Riemann hypothesis, if true, is only barely so.”

$$\Xi_t(x) = \int_0^\infty e^{tu^2}\Phi(u)\cos ux\,du$$

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If we define $F(x,t) = \Xi_t(x)$, then

$$\frac{\partial F}{\partial t} + \frac{\partial^2 F}{\partial x^2} = 0.$$

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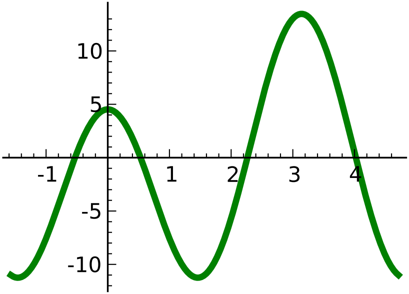
In other words $F(x,t)$ satisfies the **backwards heat equation.**

An example of something that solves the backwards heat equation:

$$f_t(x) = 10e^{4t}\cos 2x - 2\sqrt{5}e^{t}\cos x - 1$$

Movement of zeros

$$t = 0: \qquad (f_0(x) = 10\cos 2x - 2\sqrt{5}\cos x - 1)$$



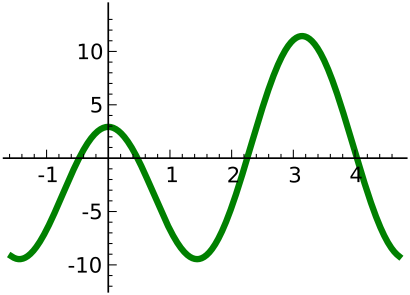
Zeros:

$$x_1, x_2 = \pm 0.532$$
$$x_3, x_4 = \pi \pm 0.879$$

As we can see, all four zeros of the original function f are real.

Movement of zeros

$t = -0.05$:



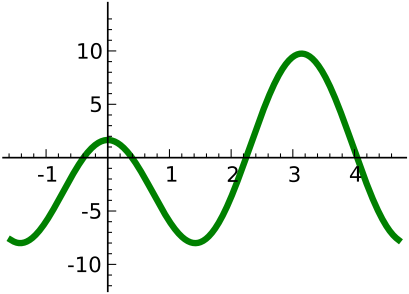
Zeros:

$$x_1, x_2 = \pm 0.473$$
$$x_3, x_4 = \pi \pm 0.889$$

As we move time back, the peaks get smaller.

Movement of zeros

$t = -0.1$:



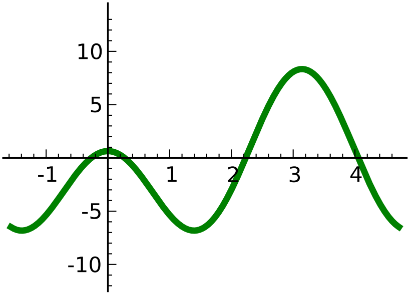
Zeros:

$$x_1, x_2 = \pm 0.393$$
$$x_3, x_4 = \pi \pm 0.900$$

As we move time back, the peaks get smaller.

Movement of zeros

$t = -0.15$:



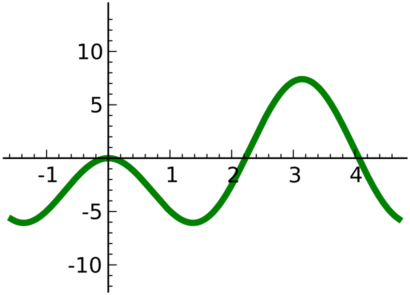
Zeros:

$$x_1, x_2 = \pm 0.269$$
$$x_3, x_4 = \pi \pm 0.911$$

As we move time back, the peaks get smaller.

Movement of zeros

$t \approx -0.188565066$:



Zeros:

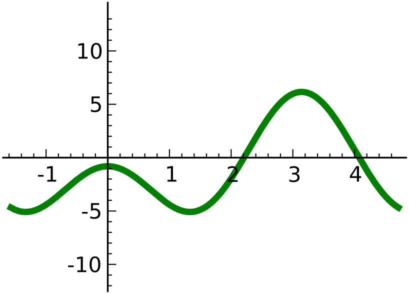
$x_1, x_2 = 0$

$x_3, x_4 = \pi \pm 0.919$

At $t \approx -0.189$, the first two zeros coalesce!

Movement of zeros

$t = -0.25$:



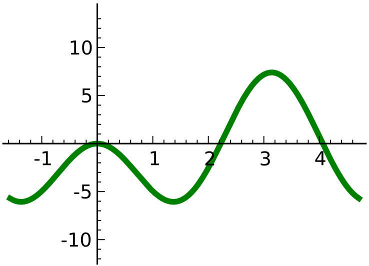
Zeros:

$x_1, x_2 = \pm 0.152i$

$x_3, x_4 = \pi \pm 0.933$

If we keep moving time back, those zeros “pop off” the real line!

$f_t(x)$ at $t_0 \approx -0.188565066$:



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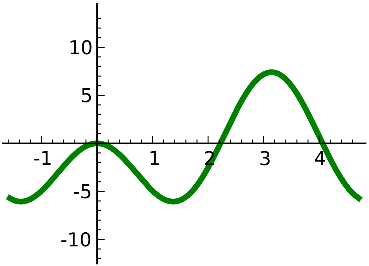

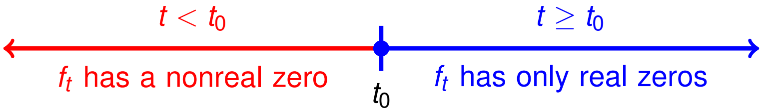

(RH: $\Lambda \leq 0$, Newman: $\Lambda \geq 0$.)

Year	Lower bound on Λ
1988	-50
1991	-5
1992	-0.39
1994	$-4.4 \cdot 10^{-6}$
2000	$-2.7 \cdot 10^{-9}$
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Strategy of Csordas, Smith, Varga (1994): look for “unusually” close pairs of zeros of $\Xi(x)$.

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Stopple investigated weaker conjecture: $\sup \Lambda_D \geq 0$.

Stopple found for $D = 175990483$, we have $-1.13 \cdot 10^{-7} < \Lambda_D$.

Possible to generalize these results even more?

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Symmetries that are not good enough:

- $\xi(s, \chi) = \xi(1 - s, \overline{\chi})$
- $\xi(s, \chi) = \epsilon\xi(1 - s, \chi)$, where $\epsilon \neq 1$.

Looking for L-functions

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Function field quadratic L-functions!

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$\mathbb{F}_q[T]$ (in "function field" setting) behaves a lot like $\mathbb{Z}$ (in "number field" setting).

As in number fields, can look at quadratic Dirichlet L-function $L(s, \chi_D)$ for fundamental discriminants $D \in \mathbb{F}_q[T]$.

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Fact: $\xi(s, \chi_D) := q^{gs} L(s, \chi_D)$ satisfies the functional equation $\xi(s, \chi_D) = \xi(1 - s, \chi_D)$. (Here, $\deg D - 1 = 2g$.)

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Bonus fact:

Theorem (RH for curves over a finite field)

If $L(s, \chi_D) = 0$, *then* $\mathrm{Re}(s) = \frac{1}{2}$.

Can define $\Xi(x, \chi_D) = \xi\left(\frac{1}{2} + i\frac{x}{\log q}, \chi_D\right)$.

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$$\begin{aligned}\Xi(x, \chi_D) &= \Phi_0 + \sum_{n=1}^{g} \Phi_n \cdot (e^{inx} + e^{-inx}) \\ &= \Phi_0 + 2\sum_{n=1}^{g} \Phi_n \cdot \cos nx\end{aligned}$$

for some $\Phi_0, \ldots, \Phi_g \in \mathbb{R}$ ($\deg D - 1 = 2g$).

$$\Xi(x, \chi_D) = \Phi_0 + 2\sum_{n=1}^{g} \Phi_n \cdot \cos nx$$

Can still follow Pólya.

$$\Xi(x, \chi_D) \xrightarrow{1} \Phi_n \xrightarrow{2} e^{tn^2}\Phi_n \xrightarrow{3} \Xi_t(x, \chi_D)$$

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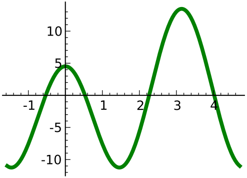
$$\Xi_t(x, \chi_D) = \Phi_0 + 2\sum_{n=1}^{g} e^{tn^2}\Phi_n \cdot \cos nx.$$

Our example from the beginning:

$$f_t(x) = 10e^{4t}\cos 2x - 2\sqrt{5}e^{t}\cos x - 1.$$



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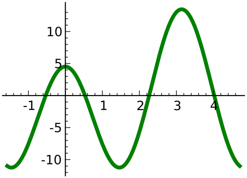
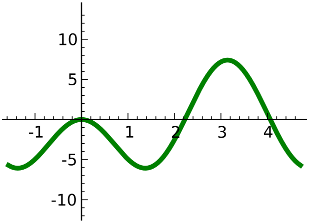


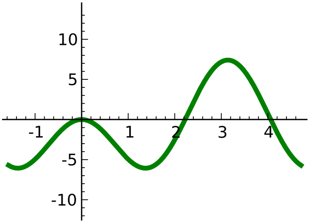
That is actually $\Xi_t(x, \chi_D)$ for

$$D = T^5 + T^4 + T^3 + 2T + 2 \in \mathbb{F}_5[T].$$



So for $D = T^5 + T^4 + T^3 + 2T + 2 \in \mathbb{F}_5[T]$,

$$\Lambda_D \approx -0.188565066.$$



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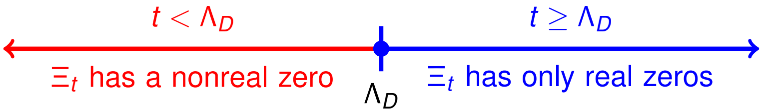

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Instead, do what Stopple did: consider an entire "family."

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Conjecture (Newman for function fields, q version)

Keep q, the size of the finite field, fixed. Then

$$\sup_{D \in \mathbb{F}_q[T]} \Lambda_D \geq 0.$$

Many different kinds of families:

Conjecture (Newman for function fields, degree version)

Keep d, the degree, fixed. Then

$$\sup_{\deg D = d} \Lambda_D \geq 0.$$

Many different kinds of families:

Conjecture (Newman for function fields, *D* version)

Fix $D \in \mathbb{Z}[T]$ squarefree. For each prime p, let D_p be the polynomial in $\mathbb{F}_p[T]$ obtained by reducing D mod p. Then

$$\sup_p \Lambda_{D_p} \geq 0.$$

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Theorem (Exact expression for Λ_{D_p})

$$\Lambda_{D_p} = \log \frac{|a_p(D)|}{2\sqrt{p}}$$

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What is the distribution of

$$\frac{a_p(D)}{2\sqrt{p}} \text{ ?}$$

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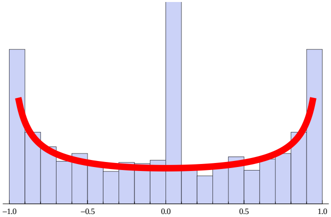

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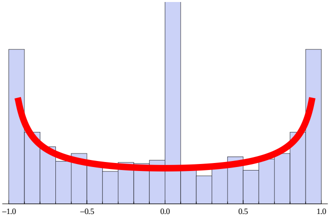

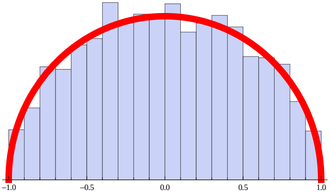

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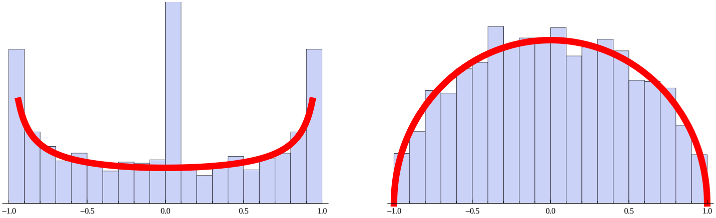
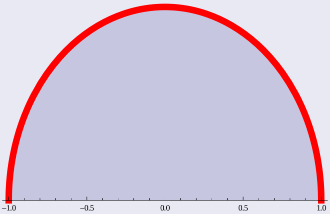


Easy to show if D has complex multiplication, then will have distribution on left.

Conjecture (Sato–Tate, 1960s)

Let $D \in \mathbb{Z}[T]$ be squarefree and such that the elliptic curve $y^2 = D(T)$ does not have complex multiplication. Then as p varies, the distribution of $\frac{a_p(D)}{2\sqrt{p}}$ is:



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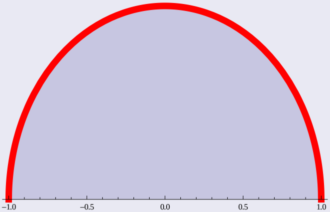

Clozel, Harris, Shepherd-Barron, and Taylor announced a proof in 2006. (They only proved for special cases.)

$$\Lambda_{D_p} = \log \frac{|a_p(D)|}{2\sqrt{p}}$$

Theorem (Newman's conjecture for fixed D, $\deg D = 3$)

Let $D \in \mathbb{Z}[T]$ be squarefree with $\deg D = 3$. If Sato-Tate is true for $y^2 = D(T)$ or if the curve has complex multiplication, then $\sup_p \Lambda_{D_p} = 0$.

$$\Lambda_{D_p} = \log \frac{|a_p(D)|}{2\sqrt{p}}$$

Theorem (Newman's conjecture for fixed D, $\deg D = 3$)

Let $D \in \mathbb{Z}[T]$ be squarefree with $\deg D = 3$. If Sato-Tate is true for $y^2 = D(T)$ or if the curve has complex multiplication, then $\sup_p \Lambda_{D_p} = 0$.

Proof.

We can find a sequence of primes $p_1, p_2, \ldots$ such that

$$\lim_{n\to\infty} \frac{a_{p_n}(D)}{2\sqrt{p_n}} \to 1.$$

□

Things to look at?

- Fix D of higher degree? (much harder)
- Study the other versions of Newman's conjecture.

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