Newman’s Conjecture in Various Settings

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What is Newman’s conjecture about?

Newman’s conjecture is related to the Riemann zeta function \( \zeta(s) \). It is an “almost counter-conjecture” to the Riemann hypothesis!
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- We’ll look at what happens when we study Newman’s conjecture in the function fields setting.
The Riemann zeta function is initially defined, for $\text{Re}(s) > 1$, by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \left( = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \right).$$

### Riemann Hypothesis (1859)

If $\zeta(s) = 0$, then either $s$ is a “trivial zero” or $\text{Re}(s) = \frac{1}{2}$. 
Define a new function $\Xi(x)$ for $x \in \mathbb{C}$ as follows:

- Let $\xi(s) = \frac{1}{2} s(s - 1) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ ("completed zeta function").
- Let $\Xi(x) = \xi \left( \frac{1}{2} + ix \right)$
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Facts:

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Facts:

- If $x \in \mathbb{R}$, then $\Xi(x) \in \mathbb{R}$.
- RH is equivalent to: all the zeros of $\Xi(x)$ are real.
Pólya’s idea (around 1920s):

$\Xi(x)$

- Step 0: Start with $\Xi(x)$
Pólya’s idea (around 1920s):

\[ \Xi(x) \xrightarrow[1]{\text{Step 0: Start with } \Xi(x)} \Phi(u) \]

- Step 0: Start with \( \Xi(x) \)
- Step 1: Take the Fourier transform

\[
\Phi(u) = \frac{1}{2\pi} \int_{0}^{\infty} \Xi(x) \cos ux \, dx.
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Pólya’s idea (around 1920s):

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Pólya’s idea (around 1920s):

\[ \Xi(x) \xrightarrow{1} \Phi(u) \xrightarrow{2} e^{tu^2} \Phi(u) \xrightarrow{3} \Xi_t(x) \]

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- **Step 2**: Multiply by \( e^{tu^2} \)
- **Step 3**: Fourier inversion
  
  \[ \Xi_t(x) = \int_{0}^{\infty} e^{tu^2} \Phi(u) \cos ux \, du. \]
In other words, study a family of functions given by

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with \( \Xi_0(x) = \Xi(x) \).

De Bruijn and Newman showed there exists \( \Lambda \in \mathbb{R} \) (called the De Bruijn–Newman constant) which divides the real line in half:

- \( t < \Lambda \) implies \( \Xi_t \) has a nonreal zero
- \( t \geq \Lambda \) implies \( \Xi_t \) has only real zeros
Newman's conjecture

**Relationship of \( \Lambda \) to RH**

\[ t < \Lambda \quad \text{\( \Xi_t \) has a nonreal zero} \]

\[ t \geq \Lambda \quad \text{\( \Xi_t \) has only real zeros} \]
Newman’s conjecture

**Relationship of \( \Lambda \) to RH**

\[
\begin{align*}
\Xi_t & \text{ has a nonreal zero} & \Lambda < \Lambda \\
\Xi_t & \text{ has only real zeros} & \Lambda \geq \Lambda
\end{align*}
\]

RH \( \iff \) \( \Xi_0 \) has only real zeros
Newman's conjecture

**Relationship of $\Lambda$ to RH**

![Diagram showing the relationship between $t$, $\Lambda$, $\Xi_t$, and RH](image)

- $t < \Lambda$ implies $\Xi_t$ has a nonreal zero.
- $t \geq \Lambda$ implies $\Xi_t$ has only real zeros.

RH $\iff$ $\Xi_0$ has only real zeros $\iff$ $\Lambda \leq 0$
Newman’s conjecture

Relationship of $\Lambda$ to RH

$\exists_t$ has a nonreal zero $\iff t < \Lambda$

$\exists_t$ has only real zeros $\iff t \geq \Lambda$

$\text{RH} \iff \exists_0$ has only real zeros $\iff \Lambda \leq 0$

Conjecture (Newman)

$\Lambda \geq 0$
Relationship of $\Lambda$ to RH

$\Xi_t$ has a nonreal zero \quad $\longleftrightarrow$ \quad $t < \Lambda$

$\Xi_t$ has only real zeros \quad $\longleftrightarrow$ \quad $t \geq \Lambda$

RH $\iff$ $\Xi_0$ has only real zeros $\iff$ $\Lambda \leq 0$

Conjecture (Newman)

$\Lambda \geq 0$

Newman: “The new conjecture is a quantitative version of the dictum that the Riemann hypothesis, if true, is only barely so.”
\[ \Xi_t(x) = \int_0^\infty e^{tu^2} \Phi(u) \cos ux \, du \]
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If we define \( F(x, t) = \Xi_t(x) \), then

\[ \frac{\partial F}{\partial t} + \frac{\partial^2 F}{\partial x^2} = 0. \]
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\]

In other words \( F(x, t) \) satisfies the \textit{backwards heat equation}. 
An example of something that solves the backwards heat equation:

\[ f_t(x) = 10e^{4t} \cos 2x - 2\sqrt{5}e^t \cos x - 1 \]
Movement of zeros

$t = 0$: \((f_0(x) = 10 \cos 2x - 2 \sqrt{5} \cos x - 1)\)

Zeros:
\[x_1, x_2 = \pm 0.532\]
\[x_3, x_4 = \pi \pm 0.879\]

As we can see, all four zeros of the original function \(f\) are real.
Movement of zeros

$t = -0.05$:

Zeros:

$x_1, x_2 = \pm 0.473$

$x_3, x_4 = \pi \pm 0.889$

As we move time back, the peaks get smaller.
Movement of zeros

\[ t = -0.1: \]

Zeros:
\[ x_1, x_2 = \pm 0.393 \]
\[ x_3, x_4 = \pi \pm 0.900 \]

As we move time back, the peaks get smaller.
Movement of zeros

\[ t = -0.15: \]

Zeros:
\[ x_1, x_2 = \pm 0.269 \]
\[ x_3, x_4 = \pi \pm 0.911 \]

As we move time back, the peaks get smaller.
Movement of zeros

\[ t \approx -0.188565066: \]

\[
\begin{align*}
Zeros: \\
x_1, x_2 &= 0 \\
x_3, x_4 &= \pi \pm 0.919
\end{align*}
\]

At \( t \approx -0.189 \), the first two zeros coalesce!
Movement of zeros

\[ t = -0.25: \]

Zeros:
\[ x_1, x_2 = \pm 0.152i \]
\[ x_3, x_4 = \pi \pm 0.933 \]

If we keep moving time back, those zeros “pop off” the real line!
$f_t(x)$ at $t_0 \approx -0.188565066$: 
$f_t(x)$ at $t_0 \approx -0.188565066$: 

$t < t_0$  
$f_t$ has a nonreal zero  

$t \geq t_0$  
$f_t$ has only real zeros
(RH: $\Lambda \leq 0$, Newman: $\Lambda \geq 0$.)

<table>
<thead>
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<tbody>
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Strategy of Csordas, Smith, Varga (1994): look for “unusually” close pairs of zeros of $\Xi(x)$. 
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Generalized Newman Conjecture: $\Lambda_D \geq 0$ for all $D$. 
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Generalized Newman Conjecture: $\Lambda_D \geq 0$ for all $D$.

Stopple investigated weaker conjecture: $\sup \Lambda_D \geq 0$.

Stopple found for $D = 175990483$, we have $-1.13 \cdot 10^{-7} < \Lambda_D$. 
For ζ and the L-functions Stopple looked at, the completed function satisfies “nicest” symmetry possible:
Possible to generalize these results even more?

For $\zeta$ and the $L$-functions Stopple looked at, the completed function satisfies "nicest" symmetry possible:

$$\xi(s, \chi_D) = \xi(1 - s, \chi_D)$$
Generalizations of Newman’s conjecture

Possible to generalize these results even more?

For $\zeta$ and the $L$-functions Stopple looked at, the completed function satisfies “nicest” symmetry possible:

$$\xi(s, \chi_D) = \xi(1 - s, \chi_D)$$

Symmetries that are not good enough:

- $\xi(s, \chi) = \xi(1 - s, \chi)$
- $\xi(s, \chi) = \epsilon \xi(1 - s, \chi)$, where $\epsilon \neq 1$. 
Looking for $L$-functions
Looking for $L$-functions!

Automorphic $L$-functions!
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Function field quadratic $L$-functions!
Overview of function fields

Let $\mathbb{F}_q$ denote the finite field with $q$ elements.
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Let \( \mathbb{F}_q[T] \) denote ring of polynomials in \( T \) with coefficients in \( \mathbb{F}_q \).

\( \mathbb{F}_q[T] \) (in “function field” setting) behaves a lot like \( \mathbb{Z} \) (in “number field” setting).
As in number fields, can look at quadratic Dirichlet $L$-function $L(s, \chi_D)$ for fundamental discriminants $D \in \mathbb{F}_q[T]$. 

Fact: 

$$\xi(s, \chi_D) := q^{gs}L(s, \chi_D)$$ satisfies the functional equation 

$$\xi(s, \chi_D) = \xi(1-s, \chi_D).$$ (Here, $\deg D - 1 = 2g$.)

Bonus fact: 

Theorem (RH for curves over a finite field) 

If $L(s, \chi_D) = 0$, then $\Re(s) = \frac{1}{2}$. 

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*If $L(s, \chi_D) = 0$, then $\Re(s) = \frac{1}{2}$.***
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$$\Xi(x, \chi_D) = \Phi_0 + \sum_{n=1}^{g} \Phi_n \cdot (e^{inx} + e^{-inx})$$

$$= \Phi_0 + 2 \sum_{n=1}^{g} \Phi_n \cdot \cos nx$$

for some $\Phi_0, \ldots, \Phi_g \in \mathbb{R}$ (deg $D - 1 = 2g$).
Newman's conjecture in function fields

\[ \Xi(x, \chi_D) = \Phi_0 + 2 \sum_{n=1}^{g} \Phi_n \cdot \cos nx \]

Can still follow Pólya.

\[ \Xi(x, \chi_D) \xrightarrow{1} \Phi_n \xrightarrow{2} e^{tn^2} \Phi_n \xrightarrow{3} \Xi_t(x, \chi_D) \]
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Important! Here we take the Fourier transform on the circle. We end up with

\[ \Xi_t(x, \chi_D) = \Phi_0 + 2 \sum_{n=1}^{g} e^{tn^2} \Phi_n \cdot \cos nx. \]
Our example from the beginning:

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That is actually \( \Xi_t(x, \chi_D) \) for

\[ D = T^5 + T^4 + T^3 + 2T + 2 \in \mathbb{F}_5[T]. \]
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$$\Lambda_D \approx -0.188565066.$$
Very important: We were able to calculate $\Lambda_D!!$
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In our example, $\Lambda_D \approx -0.188565066 < 0$. Is this surprising? (Recall for $\zeta$: RH: $\Lambda \leq 0$. Newman: $\Lambda \geq 0$.)
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Don’t want to conjecture that $\Lambda_D \geq 0$ for all $D$. 
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Don’t want to conjecture that $\Lambda_D \geq 0$ for all $D$.

Instead, do what Stopple did: consider an entire “family.”
Many different kinds of families:
Many different kinds of families:

**Conjecture (Newman for function fields, $q$ version)**

*Keep $q$, the size of the finite field, fixed. Then*

$$\sup_{D \in \mathbb{F}_q[T]} \Lambda_D \geq 0.$$
Many different kinds of families:

Conjecture (Newman for function fields, degree version)

Keep $d$, the degree, fixed. Then

$$\sup_{\deg D = d} \Lambda_D \geq 0.$$
Many different kinds of families:

**Conjecture (Newman for function fields, \(D\) version)**

Fix \(D \in \mathbb{Z}[T]\) squarefree. For each prime \(p\), let \(D_p\) be the polynomial in \(\mathbb{F}_p[T]\) obtained by reducing \(D\) mod \(p\). Then

\[
\sup_p \Lambda_{D_p} \geq 0.
\]
Fix $D \in \mathbb{Z}[T]$ squarefree with $\deg D = 3$. 
Fix $D \in \mathbb{Z}[T]$ squarefree with $\deg D = 3$. For each odd prime $p$, we can reduce $D$ to $D_p \in \mathbb{F}_q[T]$ and get the function

$$\Xi_t(x, \chi_{D_p}) = -a_p(D) + 2\sqrt{p} e^t \cos x.$$
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Note: $a_p(D)$ is called the **trace of Frobenius** of the elliptic curve $y^2 = D(T)$. 
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Note: $a_p(D)$ is called the **trace of Frobenius** of the elliptic curve $y^2 = D(T)$.

**Theorem (Exact expression for $\Lambda_{D_p}$)**

$$\Lambda_{D_p} = \log \frac{|a_p(D)|}{2\sqrt{p}}$$
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Hasse showed in the 1930s that \(|a_p(D)| < 2\sqrt{p}|.\]
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What is the distribution of

\[ \frac{a_p(D)}{2\sqrt{p}} ? \]
Pick any squarefree $D \in \mathbb{Z}[T]$. Then $a_p(D)/(2\sqrt{p})$ will give you one of two distributions:
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Easy to show if $D$ has complex multiplication, then will have distribution on left.
Conjecture (Sato–Tate, 1960s)

Let $D \in \mathbb{Z}[T]$ be squarefree and such that the elliptic curve $y^2 = D(T)$ does not have complex multiplication. Then as $p$ varies, the distribution of $\frac{a_p(D)}{2\sqrt{p}}$ is:
Conjecture (Sato–Tate, 1960s)

Let $D \in \mathbb{Z}[T]$ be squarefree and such that the elliptic curve $y^2 = D(T)$ does not have complex multiplication. Then as $p$ varies, the distribution of $\frac{a_p(D)}{2\sqrt{p}}$ is:

Clozel, Harris, Shepherd-Barron, and Taylor announced a proof in 2006. (They only proved for special cases.)
\[ \Lambda_{D_p} = \log \frac{|a_p(D)|}{2\sqrt{p}} \]

**Theorem (Newman’s conjecture for fixed \( D, \deg D = 3 \))**

Let \( D \in \mathbb{Z}[T] \) be squarefree with \( \deg D = 3 \). If Sato-Tate is true for \( y^2 = D(T) \) or if the curve has complex multiplication, then \( \sup_p \Lambda_{D_p} = 0 \).
\[ \Lambda_{Dp} = \log \frac{|a_p(D)|}{2\sqrt{p}} \]

**Theorem (Newman’s conjecture for fixed \( D, \deg D = 3 \))**

Let \( D \in \mathbb{Z}[T] \) be squarefree with \( \deg D = 3 \). If Sato-Tate is true for \( y^2 = D(T) \) or if the curve has complex multiplication, then \( \sup_p \Lambda_{Dp} = 0 \).

**Proof.**

We can find a sequence of primes \( p_1, p_2, \ldots \) such that

\[ \lim_{n \to \infty} \frac{a_{p_n}(D)}{2\sqrt{p_n}} \rightarrow 1. \]
Things to look at?

- Fix $D$ of higher degree? (much harder)
- Study the other versions of Newman’s conjecture.
Acknowledgments

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- University of Maine and organizers of the Maine–Québec Number Theory Conference.

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