The n-Level Density of Dirichlet L-Functions over $\mathbb{F}_q[T]$

Julio Andrade (julio_andrade@brown.edu)
Steven J. Miller (steven.j.miller@williams.edu)
Kyle Pratt (kyle.pratt@byu.net)
Minh-Tam Trinh (mtrinh@princeton.edu)

Department of Mathematics and Statistics
Williams College

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Background

Definition (Riemann Zeta Function)

For all s in the half-plane $\operatorname{Re} s > 1$, let $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$.

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1. ζ can be written as Euler product: $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$.
2. Exists a unique way to continue ζ to smooth function on $\mathbb{C}$. Continued function has a functional equation with symmetry at $\operatorname{Re} s = 1/2$:

$$\pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

where $\Gamma(s) = \int_{\mathbb{R}_+^\times} t^s e^{-t} \frac{dt}{t}$ is the gamma function.

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3. Fix $N \in \mathbb{N}$. A Dirichlet L-function of conductor N is an L-function of the form

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

where:

1. If n is not coprime to N, then $\chi(n) = 0$.
2. Else, χ factors through a character $(\mathbb{Z}/N\mathbb{Z})^{\times} \to \mathbb{C}^{\times}$.
3. N is the minimal positive integer such that this holds.

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Definition (n-Level Density)

Assume $L(s)$ has RH. Write zeroes in form $1/2 + i\gamma_j$. Let $\phi : \mathbb{R}^n \to \mathbb{C}$ be integrable. We define

$$D_{n,L}(\phi) := \sum_{\substack{\gamma_{j_1}, \ldots, \gamma_{j_n} \\ \text{distinct}}} \phi\left(\gamma_{j_1} \cdot \frac{\log c}{2\pi}, \ldots, \gamma_{j_n} \cdot \frac{\log c}{2\pi}\right)$$

where c is a normalization constant, the analytic conductor of L (Normalizes so that average spacing is 1).

Density Conjectures

The Katz-Sarnak Philosophy says:

1. L-functions fall into "families" $\mathcal{F}$. Given N, elements with conductor $c \asymp N$ have similar zero distributions. Form subfamily $\mathcal{F}_N$.

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Conjecture that

$$\begin{aligned} D_{n,\mathcal{F}_N}(\phi) &:= \frac{1}{\#\mathcal{F}_N} \sum_{L \in \mathcal{F}_N} D_{n,L}(\phi) \\ &\to \int_{\mathbb{R}^n} \phi(x_1, \ldots, x_n) W_{n,G}(x_1, \ldots, x_n)\, \mathrm{d}x_1 \ \ldots\ \mathrm{d}x_n \end{aligned}$$

where $W_{n,G}$ depends only on n and G.

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1. For the Riemann zeta function and families of L-functions over number fields, little is known.
2. Most results assume $\widehat{\phi}$ exists and is supported within a fixed interval, usually $[-2, +2]$. Want to remove this restriction!
3. Can say more about families over function fields.

L-Functions over $\mathbb{F}_q[T]$

1. Fix a prime power q. Define $\mathbb{F}_q[T]$ to be ring of polynomials in T over the finite field $\mathbb{F}_q$.

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3.

$\mathbb{Z} \subseteq \mathbb{Q}$	$\mathbb{F}_q[T] \subseteq \mathbb{F}_q(T)$
Positive n	Monic f
Prime p	Irreducible P
$\lvert n \rvert = \#\mathbb{Z}/n\mathbb{Z}$	$\lvert f \rvert = \#\mathbb{F}_q[T]/f\mathbb{F}_q[T] = q^{\deg f}$

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Prime p	Irreducible P
$\lvert n\rvert = \#\mathbb{Z}/n\mathbb{Z}$	$\lvert f\rvert = \#\mathbb{F}_q[T]/f\mathbb{F}_q[T] = q^{\deg f}$
$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$	$\zeta_{\mathbb{F}_q[T]}(s) = \sum_f' \lvert f\rvert^{-s}$
$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$	$L_{\mathbb{F}_q[T]}(s,\chi) = \sum_f' \chi(f) \lvert f\rvert^{-s}$

Properties of L-Functions over $\mathbb{F}_q[T]$

1. $\zeta_{\mathbb{F}_q[T]}(s)$ is very simple:

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2. $L_{\mathbb{F}_q[T]}(s, \chi)$ is a polynomial in q^{-s}, so has finitely many zeros.
3. Weil: Nontrivial zeros of $L_{\mathbb{F}_q[T]}(s, \chi)$ are

$$s = 1/2 + i\gamma_\chi$$

with $\gamma_\chi \in \mathbb{R}$. Geometric RH has been proven! (Rosen, *Number Theory in Function Fields*, p. 41)

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3. Log-conductor is number of zeroes of $L_{\mathbb{F}_q[T]}(s, \chi)$. Do not include conductor in normalization of spacings. Weight zeroes to have total mass 1.

Context

Let $Q \in \mathbb{F}_q[T]$ be irreducible of degree d. Let $\mathcal{F}_Q$ be family of Dirichlet L-functions over $\mathbb{F}_q[T]$ of conductor Q.

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1. In the limit $q \to \infty$ with d fixed, Sato-Tate equidistribution for $\mathbb{F}_q(T)$ implies that this family has unitary statistics

$$D_{1,\mathcal{F}_Q}(\phi) := \frac{1}{(d-1)\#\mathcal{F}_Q} \sum_{\chi \in \mathcal{F}_Q} \sum_{\gamma_\chi} \phi\left(\gamma_\chi / T_q\right) \to a_\phi(0)$$

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3. In studying $\mathcal{F}_Q$, we extend Katz-Sarnak to a new kind of geometric family.

1-Level Density

Theorem (A-M-P-T)

Let $Q \in \mathbb{F}_q[T]$ be irreducible of degree $d \geq 2$. Let $\mathcal{F}_Q$ be the family of Dirichlet L-functions of conductor Q. Let $\phi(s) = \sum_{n\in\mathbb{Z}} a_\phi(n)e^{2\pi ins}$ be a test function such that, for some $\epsilon > 0$,

$$a_\phi(n) \ll \frac{1}{n^{2+\epsilon}q^{n/2}} \tag{2.1}$$

for all n large enough. Then

$$D_{1,\mathcal{F}_Q}(\phi) = a_\phi(0) - \frac{1}{(d-1)(q-1)}\sum_{n\in\mathbb{Z}}\frac{a_\phi(n)}{q^{|n|/2}} + O\left(\frac{1}{\epsilon d^{1+\epsilon}q^d}\right) \tag{2.2}$$

Proof sketch.

1. Start with $\phi_0(s) = \phi(s/T_q)/(d-1)$ and $1/2 < c < 1$:

$$\sum^{\mathcal{F}_Q} \sum_{\gamma_{\chi,j}} \phi_0(\gamma_{\chi,j})$$
$$= \sum^{\mathcal{F}_Q} \frac{1}{2\pi i} \left(\int_{c-\frac{\pi i}{\log q}}^{c+\frac{\pi i}{\log q}} - \int_{1-c-\frac{\pi i}{\log q}}^{1-c+\frac{\pi i}{\log q}} \right) \frac{L'}{L}(s,\chi)\phi_0(-i(s-1/2))\,\mathrm{d}s$$

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2. Replace the integral on the line $1-c$ using the functional equation of $L(s,\chi)$.
3. Key idea: ϕ has Fourier series rather than Fourier transform.

Proof sketch, cont.

1. Pulling out Fourier coefficients of ϕ, we get a geometric "Explicit Formula." Estimate

$$\sum_{n=0}^{\infty} \sum_{\substack{\deg f = n \\ f \equiv 1 \pmod{Q}}}{}' \frac{\Lambda(f) a_\phi(n)}{q^{n/2}}$$

Think of $\Lambda(f)$ as "indicator on irreducible powers."

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2. Tempted to use Primes in Arithmetic Progressions Theorem.
3. Key idea: Replace with seemingly cruder bound that does not grow exponentially in d.

Montgomery's Conjecture for $\mathbb{F}_q[T]$

To remove restriction on $a_\phi(n)$:

Conjecture (Montgomery-A-M-P-T)

Let $Q \in \mathbb{F}_q[T]$ be irreducible of degree $d \geq 2$. Let $s = 1/2 + i\gamma_\chi$ run through zeros of $L(s, \chi) \in \mathcal{F}_Q$. Then there exists $\delta > 0$ such that

$$\sum_{\chi \neq \chi_0} \sum_{\gamma_\chi} q^{in\gamma_\chi} \ll (d-1)(\#\mathcal{F}_Q)^{1-\delta}$$

for all n.

Using Erdős-Turán, can determine conjectural error term:

Proposition (A-M-P-T)

Let $Q \in \mathbb{F}_q[T]$ be irreducible of degree $d \geq 2$. Let $s = 1/2 + i\gamma_\chi$ run through zeros of $L(s, \chi) \in \mathcal{F}_Q$. Suppose that for some $0 \leq \epsilon_1, \epsilon_2 < 1$,

$$\sum_{\chi \neq \chi_0} \sum_{\gamma_\chi} q^{in\gamma_{\chi,j}} \ll (d-1)^{1-\epsilon_1} \# \mathcal{F}_Q^{1-\epsilon_2}$$

holds for all n. Then

$$D_{1,\mathcal{F}_Q}(\phi) = a_\phi(0) + O\left(\frac{\epsilon_2 d}{d^{\epsilon_1}(\#\mathcal{F}_Q)^{\epsilon_2}}\right)$$

for all test functions $\phi(s) = \sum_{n \in \mathbb{Z}} a_\phi(n) e^{2\pi i n s}$.

2-Level Density

Theorem (A-M-P-T)

Let $Q \in \mathbb{F}_q[T]$ be irreducible of degree $d \geq 2$. Let $\phi(s_1, s_2) = \sum_{n \in \mathbb{Z}^2} a_\phi(n_1, n_2) e^{2\pi i n \cdot s}$, such that for some $\epsilon > 0$,

$$a_\phi(n_1, n_2) \ll \frac{1}{(n_1 n_2)^{2+\epsilon} q^{(n_1+n_2)/2}}$$

for all n large enough. Then

$$D_{2,\mathcal{F}_Q}(\phi) := \frac{1}{(d-1)^2 \# \mathcal{F}_Q} \sum_{\chi \in \mathcal{F}_Q} \sum_{\gamma_{\chi,1} \neq \gamma_{\chi,2}} \phi\left(\frac{\gamma_{\chi,1}}{T_q}, \frac{\gamma_{\chi,2}}{T_q}\right)$$
$$\to a_\phi(0,0) - a_{\phi_{\text{diag}}}(0)$$

as $d \to \infty$, where $\phi_{\text{diag}}(s) = \phi(s, s)$. The rate of convergence can be made explicit.

Future Directions

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2. Andrade is developing similar results for the hyperelliptic ensemble, conjectured to have symplectic statistics.

Ratios Conjecture

1. Tool for handling n-level density is the Ratios Conjecture. Estimate
$$\sum_{L \in \mathcal{F}} \prod_{j=1}^{n} \frac{L(1/2+\alpha_j)}{L(1/2+\gamma_j)}$$
using approximate functional equation and "illegal" averaging heuristics.

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2. To compute n-level density from Ratios Conjecture:
   1. Differentiate estimate w.r.t. $\alpha_1, \dots, \alpha_n$ and set $\gamma_j = \alpha_j$. Error is still small.
   2. Again, contour integration!

Conrey-Snaith and Mason stated Ratios for number-field families:

1. Shifts of $\zeta(s)$.
2. Quadratic Dirichlet L-functions $L(s, \chi_D)$ with $D \leq N$.
3. Quadratic Dirichlet twists of fixed $L(s, E)$ of even sign.

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Conjecture (A-M-P-T)

Let $Q \in \mathbb{F}_q[T]$ be irreducible of degree ≥ 2. Choose $\alpha_j, \beta_j, \gamma_j, \delta_j$ in the region $\{s : |\operatorname{Re} s| < 1/4$ and $|\operatorname{Im} s| \ll |Q|\}$. Then

$$\sum_{\chi \neq \chi_0} \prod_{j=1}^{n} \frac{L(1/2+\alpha_j,\chi)L(1/2+\beta_j,\overline{\chi})}{L(1/2+\gamma_j,\chi)L(1/2+\delta_j,\overline{\chi})}$$

$$= \#\mathcal{F}_Q \cdot \prod_{j,k=1}^{n} \frac{(1-q^{-(\alpha_j+\delta_k)})(1-q^{-(\beta_j+\gamma_k)})}{(1-q^{-(\alpha_j+\beta_k)})(1-q^{-(\gamma_j+\delta_k)})} A_{\mathcal{F}_Q} + O(|Q|^{1/2+\epsilon})$$

for all $\epsilon > 0$, where $A_{\mathcal{F}_Q} = A_{\mathcal{F}_Q}(\alpha,\beta,\gamma,\delta) \approx 1$ is an Euler product.

FINIS

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