Distribution of Summands in Generalized Zeckendorf Decompositions

Archit Kulkarni, Carnegie Mellon University
Umang Varma, Kalamazoo College
auk@andrew.cmu.edu, Umang.Varma10@kzoo.edu

(with Philippe Demontigny, Thao Do, and David Moon)

Advisor: Steven J Miller
SMALL REU, Williams College

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Zeckendorf's Theorem

Our research is inspired by an elegant theorem of Zeckendorf.

Theorem

Write the Fibonacci numbers as $F_1 = 1$, $F_2 = 2$, $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. All natural numbers can be uniquely written as a sum of non-consecutive Fibonacci numbers.

Proof of existence uses the greedy algorithm and uniqueness follows by induction.

Example

$$2013 = 1597 + 377 + 34 + 5 = F_{16} + F_{13} + F_8 + F_4$$

Previous research

- The number of summands in the Zeckendorf decomposition of integers in $[F_n, F_{n+1})$ converges to the Gaussian distribution as $n \to \infty$.
- This result extends to non-negative recurrence relations of the form

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1},$$

where $c_1, c_2, \ldots, c_L$ are nonnegative integers and L, c_1, c_L are positive.

Going the other way

Previous work: linear recurrence sequence $\rightarrow$ notion of legality.

Our work: notion of legality $\rightarrow$ linear recurrence sequence.

f-Decompositions

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Definition

Let $f : \mathbb{N}_0 \to \mathbb{N}_0$. A sum $x = \sum_{i=0}^{k} a_{n_i}$ of terms of $\{a_n\}$ is an *f-decomposition of x using $\{a_n\}$* if for every a_{n_i}, the previous $f(n_i)$ terms are not in the sum.

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Example (Base b representation)

Base b representation can be interpreted as f-decompositions. For example, consider base 5:

$$\{a_n\} = \underbrace{1, 2, 3, 4}, \underbrace{5, 10, 15, 20}, \underbrace{25, 50, 75, 100}, \dots$$

Here, $a_n = 5a_{n-4}$.

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Example (Zeckendorf Decompositions)

Consecutive terms may not be chosen: this is equivalent to saying $f(n) = 1$ for all $n \in \mathbb{N}$.
We construct a sequence by appending the smallest number that cannot be decomposed using previous terms.

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$$\{F_n\} = 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

Analogue to Zeckendorf's Theorem

Theorem

For any $f : \mathbb{N}_0 \to \mathbb{N}_0$*, there exists a sequence of natural numbers* $\{a_n\}_{n=0}^{\infty}$ *such that every positive integer has a unique* f*-decomposition using* $\{a_n\}$*.*

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Construction of $\{a_n\}$.

- Let $a_0 = 1$ and $a_n = a_{n-1} + a_{n-1-f(n-1)}$ (assume $a_n = 1$ when $n < 0$). The proof of existence is by induction.

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- Consider any $x \in [a_m, a_{m+1}) = [a_m, a_m + a_{m-f(m)})$. Clearly, $x - a_m \in [0, a_{m-f(m)})$.

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- Consider any $x \in [a_m, a_{m+1}) = [a_m, a_m + a_{m-f(m)})$. Clearly, $x - a_m \in [0, a_{m-f(m)})$.
- By the induction hypothesis, $x - a_m$ has an f-decomposition in $\{a_n\}_{n=0}^{m-f(m)-1}$.

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- Using linear algebra, we can show that the subsequence $\{a_{i,n}\} = a_i, a_{i+b}, a_{i+2b}, a_{i+3b}, \dots$ satisfies some linear recurrence relation for all $i \in \{0, 1, 2, \dots, b-1\}$.

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- By finding a common recurrence relation we can show that the interleaved sequence satisfies a linearly recurrent sequence.

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- $\{F_n\} = 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$ satisfies $F_n = F_{n-1} + F_{n-2}$.

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  satisfies $F_n = F_{n-1} + F_{n-2}$.
- $\{a_n\} = 1, 4, 15, 56, 209, 780, 2911, 10864, 40545, \dots$
  satisfies $a_n = 4a_{n-1} - a_{n-2}$.

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- $p_F(x)p_a(x) = (x^2-x-1)(x^2-4x+1) = x^4-5x^3+4x^2+3x-1$.

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- $p_F(x)p_a(x) = (x^2 - x - 1)(x^2 - 4x + 1) = x^4 - 5x^3 + 4x^2 + 3x - 1$.
- Both sequences satisfy $s_n = 5s_{n-1} - 4s_{n-2} - 3s_{n-3} + s_{n-4}$.

$$\begin{aligned}(s_n - s_{n-1} - s_{n-2}) - 4(s_{n-1} - s_{n-2} - s_{n-3}) \\ + (s_{n-2} - s_{n-3} - s_{n-4}) &= 0\end{aligned}$$

Motivation

Previous research only handles linear recurrence relations with nonnegative coefficients.

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Answer

Yes! Some functions f yield sequences $\{a_n\}$ that can only be described by linear recurrence relations with at least one negative coefficient.

Definition of 3-Bin Decomposition

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We have "bins" of width 3, and a legal decomposition contains no consecutive terms and at most one term from each bin.

$$\{a_n\} = \underbrace{1, 2, 3}, \underbrace{4, 7, 11}, \underbrace{15, 26, 41}, \underbrace{56, 97, 153}, \underbrace{209, 362, 571}, \ldots$$

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We can show that no linear recurrence relation describing this sequence has all nonnegative coefficients, and that the shortest such relation is $a_n = 4a_{n-3} - a_{n-6}$. Thus, these decompositions are out of the scope of previous work.

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Question

Consider the number of summands for an integer randomly chosen from the interval $[0, a_{3n})$. As $n \to \infty$, what can we say about the distribution?

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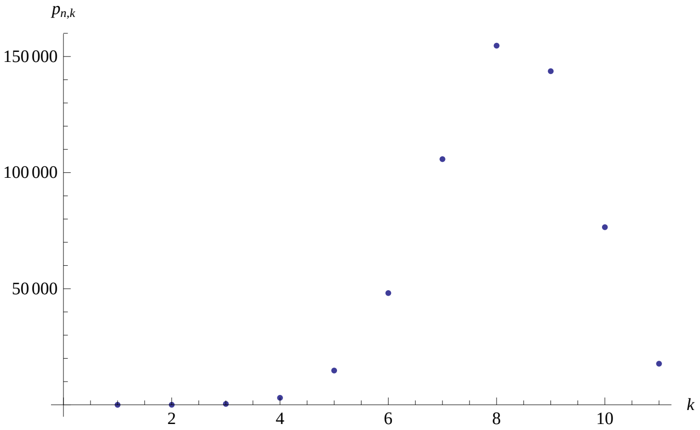
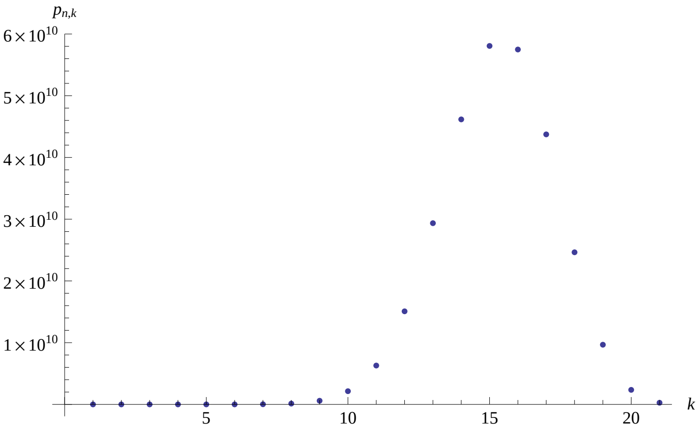
For positive linear recurrence sequences, previous work has shown that the distribution approaches a Gaussian. Can we extend this to our new sequences?

Gaussian Behavior of the Number of Summands



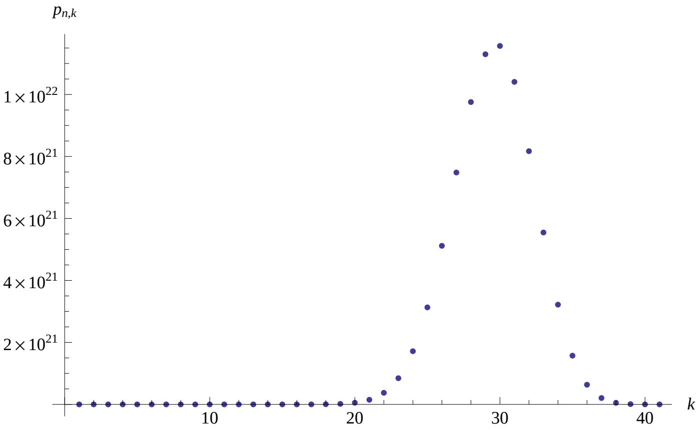
First 10 bins

Gaussian Behavior of the Number of Summands



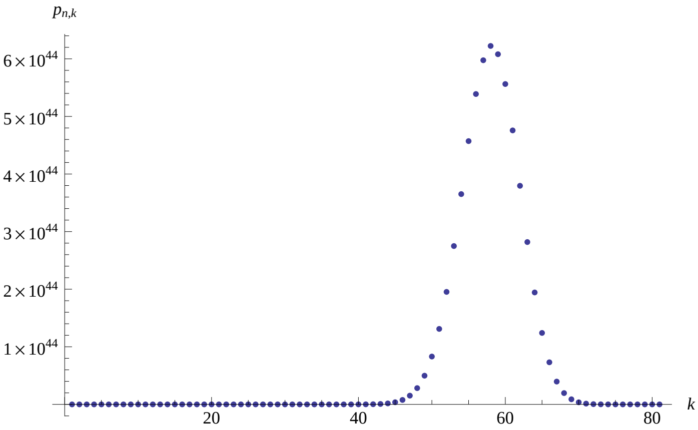
First 20 bins

Gaussian Behavior of the Number of Summands



First 40 bins

Gaussian Behavior of the Number of Summands



First 80 bins

Number of summands

Let $p_{n,k}$ be the number of integers whose decomposition uses k summands all from the first n bins.

Claim

$p_{n,k} = p_{n-1,k} + 3p_{n-1,k-1} - p_{n-2,k-2}$

We count the number of ways to choose the summands.

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- One of the three summands is chosen from the first bin.
- Can't choose both the last element of the first bin and the first element of the second bin.

Generating Function

- Begin with

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- Let $F(x,y) = \sum_{n,k\geq 0} p_{n,k} x^n y^k$. We can write

$$F(x,y) = xF(x,y) + 3xyF(x,y) - x^2y^2F(x,y) + 1$$

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- Define $g_n(y) = \sum_{k \geq 0} p_{n,k} y^k$ to isolate x^n terms. Using partial fractions and Taylor series,

$$g_n(y) = \frac{\left(3y+1+\sqrt{5y^2+6y+1}\right)^{n+1} - \left(3y+1-\sqrt{5y^2+6y+1}\right)^{n+1}}{2^{n+1}\sqrt{5y^2+6y+1}}$$

Mean and Variance

- We can use $g_n(y) = \sum_{k\geq 0} p_{n,k} y^k$ to compute mean and variance of the random variable X_n, the number of summands for an integer randomly chosen from $[0, a_{3n})$.

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- The mean is

$$\mu_n = \frac{g_n'(1)}{g_n(1)} = \frac{8+3\sqrt{12}}{\sqrt{12}\left(4+\sqrt{12}\right)} n + O(1)$$

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- The variance is

$$\sigma_n^2 = \frac{\frac{d}{dy}\left[y g_n'(y)\right]\Big|_{y=1}}{g(1)} - \mu^2 = \frac{\sqrt{3}}{9} n + O(1)$$

Moment Generating Function

- If we normalize X_n to $Y_n = (X_n - \mu_n)/\sigma_n$, the moment generating function of Y_n is

$$M_{Y_n}(t) = \mathbb{E}(e^{tY_n}) = \sum_{k \geq 0} \frac{p_{n,k} e^{t\frac{(k-\mu_n)}{\sigma_n}}}{\sum_{k \geq 0} p_{n,k}} = \frac{g_n(e^{t/\sigma_n}) e^{-t\mu_n/\sigma_n}}{g_n(1)}$$

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- After multiple Taylor Series expansions, we get

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- Hence, the distribution of Y_n converges to the standard normal distribution.

Generalizations

This method can easily be generalized for large classes of f-decompositions.

We have proved Gaussian behavior in the following cases:

- every bin is of width b for any $b \geq 3$
- bins have alternating widths x and y

Future Directions

- Does Gaussian behavior hold for all f-decompositions if f is periodic?
- Can we find interesting notions of legal decomposition that are more general than f-decompositions?

Thanks

We would like to thank

- Our advisor, Prof. Steven Miller.
- Our co-authors Philippe Demontigny, Thao Do, and David Moon.
- Everyone involved with the SMALL REU at Williams College.
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