Complex Ramsey Theory

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Arithmetic and Geometric Progressions

Definition

A 3-**term arithmetic progression** is a sequence of natural numbers of the form $(x, x+n, x+2n)$ where n is a positive integer.

Definitions

A 3-**term geometric progression** is a sequence of natural numbers of the form (x, xr, xr^2) where $r > 1$ is an integer. We refer to r as the **common ratio** of the sequence.

Definitions

Asymptotic Density

The **density** of a set $A \subseteq \mathbb{N}$ is defined to be

$$d(A) = \lim_{n\to\infty} \frac{|A \cap \{1, \ldots, n\}|}{n}$$

if this limit exists.

Upper Asymptotic Density

The **upper density** of a set $A \subseteq \mathbb{N}$ is defined to be

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \{1, \ldots, n\}|}{n}.$$

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Rankin's Greedy Set

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$$\prod_p \frac{p-1}{p} \prod_{i=1}^{\infty} \left(1 + \frac{1}{p^{3^i}}\right) = \frac{1}{\zeta(2)} \prod_{i=1}^{\infty} \frac{\zeta(3^i)}{\zeta(2 \cdot 3^i)} \approx 0.72.$$

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Onto the Gaussian Integers

Definition

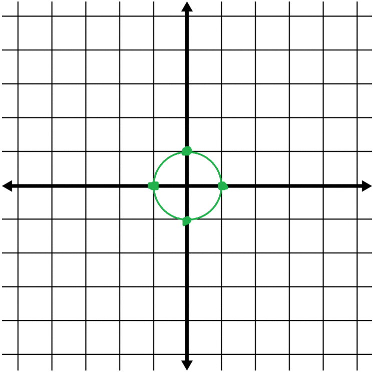
The **Gaussian integers** are defined to be the set of all $a + bi$, where a and b are integers.

Definition

The **norm** of a Gaussian integer $a + bi$ is defined to be

$$N(a + bi) = a^2 + b^2$$

Defining the Greedy Set



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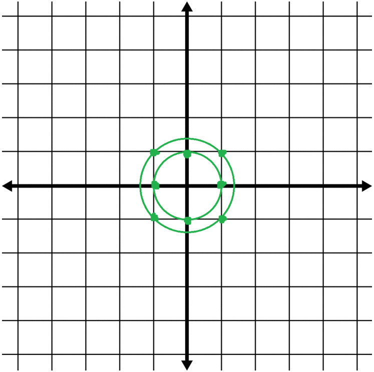

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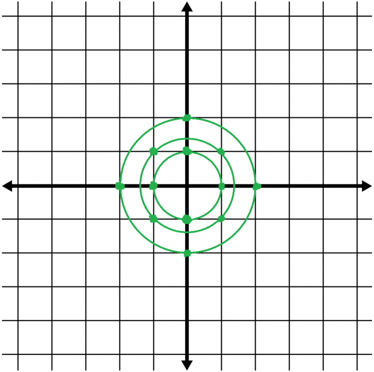

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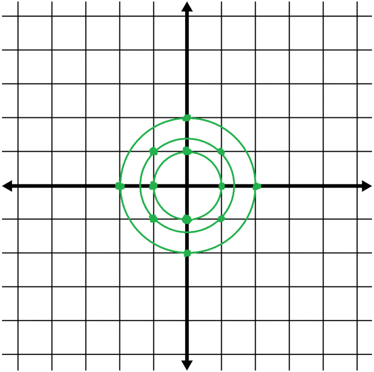
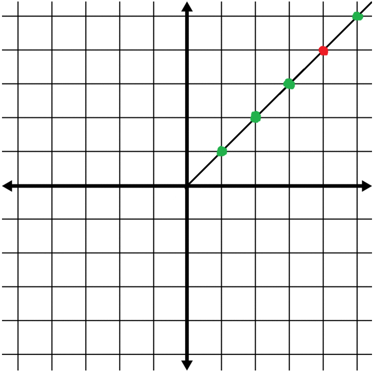


The greedy set is defined by consideration of “norm circles” whose radii increase.

Having defined it, we consider geometric progressions which avoid various kinds of ratios.

Avoiding Integral Ratios



This case can be thought of as a projection of the integral greedy set onto every line through the origin.

Depicted is the progression $1 + i, 2 + 2i, 4 + 4i$.

Avoiding Integral Ratios

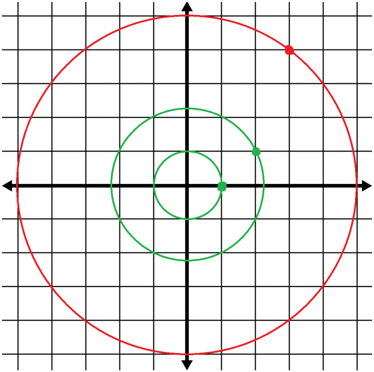
We exclude a Gaussian integer $a+bi$ exactly when it can be written in the form $k(c+di)$, where k is not in Rankin's set and $(c,d)=1$.

Theorem 1 [B,H,Mc,Mi,P,T,W '14]

The density of the greedy set of Gaussian integers that avoids integral ratios is

$$\prod_p \left(\frac{p^2-1}{p^2} \prod_{i=0}^{\infty} \left(1 + \frac{1}{p^{2\cdot 3^i}} \right) \right) = \frac{1}{\zeta(4)} \prod_{i=1}^{\infty} \frac{\zeta(2\cdot 3^i)}{\zeta(4\cdot 3^i)} \approx 0.9397.$$

Avoiding Gaussian Ratios



We also consider sets that avoid progressions with Gaussian integer ratios.

Depicted is the progression $1, 2+i, 3+4i$.

Density of the Gaussian Greedy Set

We can determine the likelihood of a Gaussian integer being included by evaluating the primes in its prime factorization and whether each prime is raised to an appropriate power.

Theorem 2 [B,H,Mc,Mi,P,T,W '14]

$$\text{Let } f(x) = \left(1 - \frac{1}{x}\right) \prod_{i=0}^{\infty} \left(1 + \frac{1}{x^{3^i}}\right).$$

Then the density of the greedy set of Gaussian integers that avoids Gaussian integral ratios is

$$f(2) \left(\prod_{p \equiv 1 \bmod 4} f^2(p) \right) \left(\prod_{q \equiv 3 \bmod 4} f(q^2) \right) \approx 0.771.$$

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An Upper Bound for Upper Density

We find an upper bound for the upper density by generalizing an argument by Riddell (1969). Looking at the subset of Gaussian integers with norm $\leq M$, we see

For $b, r \in \mathbb{Z}[i]$ with $N(b) \leq \frac{M}{4}$ and $N(r) = 2$, the terms b, rb, r^2b have norm $\leq M$ and will always be in geometric progression.

With $N(b)$ odd we know there will be no overlap amongst chosen progressions.

An Upper Bound for Upper Density

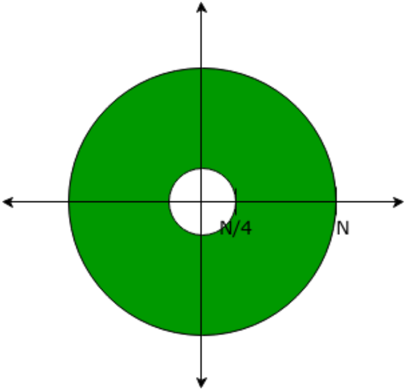
Using Gauss' circle problem, we can exclude about $\frac{1}{2} \cdot \frac{1}{2^2}$ terms. Looking at our next non-overlapping sequence (r^3b, r^4b, r^5b with $N(b) \leq \frac{M}{32}$) and continually repeating this process gives us an upper bound.

Theorem 3 [B,H,Mc,Mi,P,T,W '14]

An upper bound for the upper density is given by

$$1 - \frac{1}{2^3}\sum_{n=0}^{\infty}\frac{1}{2^{3n}} = 1 - \frac{\frac{1}{2^3}}{1-\frac{1}{2^3}} = \frac{6}{7} \approx 0.857$$

A Lower Bound for Upper Density



Generalizing an argument by McNew, we see that if we take the Gaussian integers with norm between $N/4$ and N, no three of these elements will comprise a 3-term geometric progression.

A Lower Bound for Upper Density

Similarly, we can include integers with norm between $N/16$ and $N/8$ without introducing a progression, and continue in this fashion.

Theorem 4 [B,H,Mc,Mi,P,T,W '14]

A set of acceptable norms is

$$\left(\frac{N}{25}, \frac{N}{20}\right] \cup \left(\frac{N}{16}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$$

The density of the Gaussian integers that fall inside this set gives us a lower bound of 0.8225.

Overview of Bounds

- A lower bound for maximal density of sets of Gaussian integers avoiding integral ratios is 0.9397.
- A lower bound for maximal density of sets of Gaussian integers avoiding Gaussian ratios is 0.771.
- Bounds for upper density for sets S of Gaussian integers avoiding Gaussian ratios are $0.8225 < \overline{d}(S) < 0.857$.

Future Work

- Improve the bounds on upper density for sets S of Gaussian integers avoiding Gaussian ratios.
- Define and analyze the greedy set in other number fields.
- Determine how density of maximal geometric progression-avoiding sets depend on norm and class number of other number fields.

References

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Thank you!

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