Benfordness of Zeckendorf Decompositions

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Table of Contents

Elementary Facts

Fibonacci Recurrence

$$F_{n+1} = F_n + F_{n-1}$$
$$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$$

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Binet's Formula

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

A Beautiful Theorem

Theorem (Zeckendorf - 1939)

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Example

$$\begin{aligned} 12 &= 8 + 3 + 1 \\ &= F_5 + F_3 + F_1 \\ 2016 &= 1597 + 377 + 34 + 8 \\ &= F_{16} + F_{13} + F_8 + F_5 \end{aligned}$$

Counting the Summands

Theorem (Lekkerkerker - 1952)

The average number of Fibonacci summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to

$$\frac{n}{\varphi^2 + 1} \approx .276n \qquad (1)$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

An Inquiry

Question

For a given set of numbers, how often would you expect a leading digit of 1, 2, 3, 4, . . . , 9 to occur?

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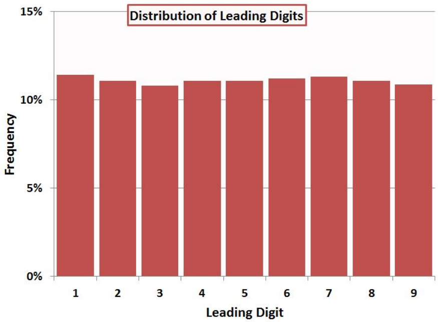

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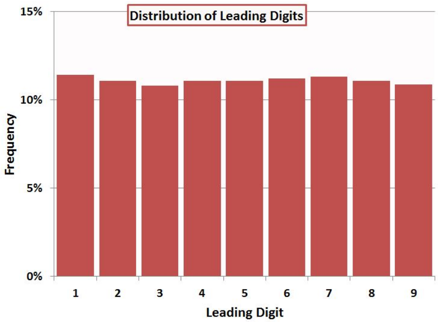
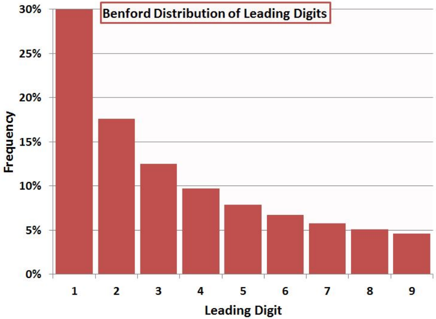


Not Quite!

The Benford Distribution

In fact, it's more like this!



Note

A leading digit 1 occurs with 30% frequency, while a leading digit 9 occurs with only 4.5 % frequency.

History of Benford's Law

- Benford's law is named after physicist Frank Benford in 1938.
- Although it was discovered earlier by Simon Newcomb in 1881.

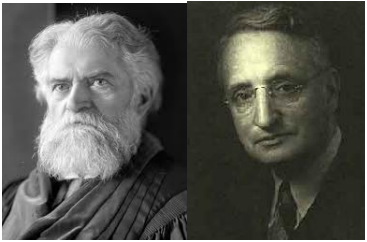


Figure: Newcomb and Benford

Benford's law

Definition (Benford's Law in Arbitrary Base)

A dataset is said to follow Benford's Law (base B) if the probability of observing a first digit of d is

$$\log_B\left(1+\frac{1}{d}\right)$$

Example

$$\mathbb{P}(d=1)=\log_{10}\left(1+\frac{1}{1}\right)\approx 0.301$$

$$\mathbb{P}(d=9)=\log_{10}\left(1+\frac{1}{9}\right)\approx 0.045$$

First Digit Bias

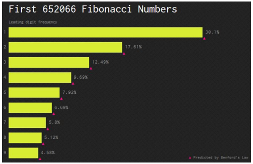


Figure: Fibonacci numbers

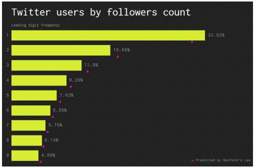


Figure: # Twitter followers

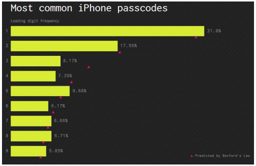


Figure: iPhone passwords

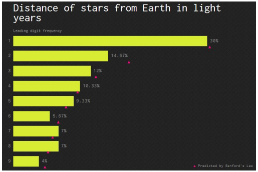


Figure: Distance of stars from Earth

Benford's Law: Applications

- $3x + 1$ problem.
- Analyzing round-off errors.
- Determining the optimal way to store numbers.
- Detecting tax fraud and data integrity.

Known Results

- Fibonacci numbers are Benford.
- Not every recurrence relation is Benford.

Example

$$a_{n+2} = 2a_{n+1} - a_n, \text{ with } a_1 = a_2 = 1$$

Sequence: $\{1, 1, 1, 1, 1, 1, 1, 1, 1, \dots\}$ is definitely not Benford.

New Results: SMALL 2014

- **Benfordness in Interval**: The distribution of the summands in the Zeckendorf decompositions, averaged over the entire interval $[F_n, F_{n+1})$.
- **Random Decomposition**: If we choose each Fibonacci number with probability q, disallowing the choice of two consecutive Fibonacci numbers, the resulting sequence follows Benford's law.
- **Benfordness of Decomposition**: If we pick a random integer M in the interval $[0, F_{n+1})$, then as $n \to \infty$, its Zeckendorf decomposition will follow Benford's Law with high probabiltiy.

Table of Contents

Theorem 1 (SMALL 2014): Benfordness in Interval

The distribution of the summands in the Zeckendorf decompositions, averaged over the entire interval $[F_n, F_{n+1})$, follows Benford's Law.

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Example

Looking at the interval $[F_5, F_6) = [8, 13)$

$$\begin{aligned} 8 &= 8 && = F_5 \\ 9 &= 8 + 1 && = F_5 + F_1 \\ 10 &= 8 + 2 && = F_5 + F_2 \\ 11 &= 8 + 3 && = F_5 + F_3 \\ 12 &= 8 + 3 + 1 && = F_5 + F_3 + F_1 \end{aligned}$$

Proof of Theorem 1

Let S be a subset of the Fibonacci numbers. Let $q(S,n)$ be the density of S over the Fibonacci numbers in the interval $[1, F_n]$. That is

$$q(S,n) = \frac{\#\{F_j \in S \mid 1 \leq j \leq n\}}{n}$$

In the case that $\lim_{n\to\infty} q(S,n)$ exists, we define the asymptotic density $q(S)$ as

$$q(S) = \lim_{n\to\infty} q(S,n)$$

Let S_d be the subset of the Fibonacci numbers which share a fixed digit d where $1 \leq d < B$. It is a well-known result that

Theorem

$$q(S_d) = \lim_{n\to\infty} q(S_d, n) = \log_B\left(1 + \frac{1}{d}\right)$$

i.e. the Fibonacci numbers are Benford!

Consider integers in the interval $I_n := [F_n, F_{n+1})$. Let $X(I_n)$ be a random variable, which takes Fibonacci numbers in $[F_1, F_n)$, weighted by how often they occur in decompositions in I_n. Then,

$$P\{X(I_n) = F_k\} := \begin{cases} \frac{F_{k-1}F_{n-k-2}}{\mu_n F_{n-1}}, & \text{if } 1 \leq k \leq n-2 \\ \frac{1}{\mu_n}, & \text{if k = n} \\ 0, & \text{otherwise} \end{cases}$$

where μ_n is the average number of summands in Zeckendorf decompositions of integers in the interval $[F_n, F_{n+1})$.

We compute an approximation for $P\{X(I_n) = F_k\}$ when $1 \leq k \leq n-2$.

$$P\{X(I_n) = F_k\} = \frac{1}{\mu_n \phi \sqrt{5}} + O\left(\phi^{-2k} + \phi^{-2n+2k}\right)$$

Then, for a fixed integer r, we compute the approximation

$$\sum_{r<k<n-r} P\{X(I_n) = F_k\} = 1 - r \cdot O\left(\frac{1}{n}\right)$$

Noting that the values of r for which the sum is vacuous do not hurt our estimates.

Set $r := \left\lfloor \frac{\log n}{\log \phi} \right\rfloor$. With these estimates, we may now compute the density of S over the Zeckendorf summands in the interval $I_n = [F_n, F_{n+1})$

$$P\{X(I_n) \in S\} = \frac{nq(S)}{\mu_n \phi \sqrt{5}} + o(1)$$

In the limit, we have

$$\lim_{n \to \infty} P\{X(I_n) \in S\} = q(S)$$

Remark

- Stronger result than Benfordness of Zeckendorf summands.
- Global property of the Fibonacci numbers can be carried over locally into the Zeckendorf summands.
- If we have a subset of the Fibonacci numbers S with asymptotic density $q(S)$, then the density of the set S over the Zeckendorf summands will converge to this asymptotic density.

Table of Contents

Theorem 2 (SMALL 2014): Random Decomposition

If we choose each Fibonacci number with probability q, disallowing the choice of two consecutive Fibonacci numbers, the resulting sequence follows Benford's law.

Example: n = 10

$$F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7 + F_8 + F_9 + F_{10}$$
$$= 2 + 8 + 21 + 89$$
$$= 120$$

Choosing a Random Decomposition

Select a random subset A of the Fibonaccis in the following way. Given some $q \in (0, 1)$. Let $A_0 := \emptyset$.

For $n \geq 1$, if $F_{n-1} \in A_{n-1}$, let $A_n := A_{n-1}$. Otherwise, let A_n equal $A_{n-1} \cup \{F_n\}$ with probability q and A_{n-1} with probability $1 - q$.

Let $A := \bigcup_n A_n$.

Goal

We will prove that, with probability 1, A is Benford.

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Stronger claim: For any subset S of the Fibonaccis with density d in the Fibonaccis, with probability 1, $S \cap A$ will have density d in A.

The probability that $F_k \in A$

The probability that $F_k \in A$ is

Lemma

$$p_k = \frac{q}{1+q} + O(q^k).$$

Expected Value of X_n

Define $X_n := \#A_n$. Using elementary techniques, we get

Lemma

$$E[X_n] = \frac{nq}{1+q} + O(1)$$

Expected Value of X_n

$X_n := \#A_n$. Let

$$x_k := \begin{cases} 1 & \text{if} F_k \in A \\ 0 & \text{if} F_k \notin A. \end{cases}$$

Expected Value of X_n

$X_n \ := \ \#A_n$. Let

$$x_k \ := \ \begin{cases} 1 & \text{if} F_k \in A \\ 0 & \text{if} F_k \notin A. \end{cases}$$

And note that $X_n := \sum_{k=1}^{n} x_k$. Then

$$\begin{aligned} E[X_n] \ &= \ \sum_{k=1}^{n} E[x_k] \\ &= \ \sum_{k=1}^{n} p_k \\ &= \ \frac{nq}{1+q} + O(1). \end{aligned}$$

Variance of X_n

$X_n := \#A_n$. Via standard calculations, we get

Lemma

$$Var[X_n] = O(n).$$

By Chebyshev's inequality, we deduce that

Corollary

$$\begin{aligned} X_n &= E[X_n] + o(n) \\ &= \frac{nq}{1+q} + o(n) \end{aligned}$$

with probability $1 + o(1)$.

Expected Value of Y_n

Define $Y_{n,S} := \#A_n \cap S$. Using standard techniques, we get

Lemma

$$E[Y_n] = \frac{nqd}{1+q} + o(n).$$

Variance of $Y_{n,S}$

$$Y_{n,S} := \#A_n \cap S$$

Lemma

$$Var[Y_{n,S}] = o(n^2)$$

Corollary

$$Y_{n,S} = \frac{nqd}{1+q} + o(n)$$

with probability $1 + o(1)$

Lemma

$$\lim_{n\to\infty} \frac{Y_{n,S}}{X_n} = \lim_{n\to\infty} \frac{\#A_n \cap S}{\#A_n} = \lim_{n\to\infty} \frac{\frac{nqd}{1+q} + o(n)}{\frac{nq}{1+q} + o(n)} = d$$

with probability 1.

But by definition, this means that $A \cap S$ has density d in A. Therefore, our claim is proven.

Table of Contents

Theorem 3 (SMALL 2014): Benfordness of Decomposition

If we pick a random integer M in the interval $[0, F_{n+1})$, then as $n \to \infty$, its Zeckendorf decomposition will follow Benford's Law with high probabiltiy.

Proof of Theorem 3

Let M be an integer in $[0, F_{n+1})$ with decomposition $M = F_{a_1} + F_{a_2} + \cdots + F_{a_\ell}$. Then the probability that $M = \sum_{F_k \in A_n} F_k$

is

$$p_M = \begin{cases} q^\ell (1-q)^{n-2\ell} \text{if} & a_\ell \leq n \\ q^\ell (1-q)^{n-2\ell+1} \text{if} & a_\ell = n \end{cases}$$

Choosing $q = \frac{1}{\varphi^2}$, the previous formula simplifies to

Lemma

$$p_M = \begin{cases} \varphi^{-n} \text{if} & M \in [0, F_n) \\ \varphi^{-n-1} \text{if} & M \in [F_n, F_{n+1}) \end{cases}$$

We can prove that when selecting integers from $[0, F_{n+1})$ uniformly at random, for any $\varepsilon > 0$ the proportion of the summands of M which are in S will be within ε of S with probability $1 + o(1)$.

Table of Contents

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Appendix

We know that the probability of choosing $M = F_{a_1} + F_{a_2} + \cdots + F_{a_\ell}$ is

p_M equals $q^\ell(1-q)^{n-2\ell}$ if $a_\ell \leq n$ and $q^\ell(1-q)^{n-2\ell+1}$ if $a_\ell = n$.

Choosing $q = \frac{1}{\varphi^2}$ gives

p_M equals φ^{-n} if $M \in [0, F_n)$ and φ^{-n-1} if $M \in [F_n, F_{n+1})$.

Let $e_n(x) = \left|\frac{\#D_M \cap S}{D_M} - d\right|$, where D_M is the decomposition of M. Let $E = \{M \in [0, F_{n+1}) : e(x) \geq \varepsilon\}$.
Let $B_n = \sum_{F_k \in A_n} F_k$, so that $p(M) = P(B_n = M)$. For sufficiently large n, we have $F_{n+1} \geq \frac{\phi^{n+1}}{\sqrt{5}}$. Now let M_n be a random variable selected uniformly at random from the integers in $[0, F_{n+1})$. Then

$$P(M_n \in E_n) = \frac{\#E_n}{F_{n+1}} = \sum_{M \in E_n} \frac{1}{F_{n+1}} \tag{2}$$

$$\leq \sqrt{5} \sum_{M \in E_n} \phi^{-n-1} \leq \sqrt{5} \sum_{M \in E_n} p(M) \tag{3}$$

$$= \sqrt{5} P(B_n \in E_n) = o(1) \tag{4}$$

Therefore, we have proved that when selecting integers from $[0, F_{n+1})$ uniformly at random, for any $\varepsilon > 0$ the proportion of the summands of x which are in S will be within ϵ of S with probability $1 + o(1)$.