Maine-Québec Number Theory Conference: A Shifted Twisted Second Moment and Gaps Between Zeros for L-functions Associated to Holomorphic Cusp Forms

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Automorphic L-Functions

Let f be a modular cusp form of weight κ and level q, f has a (normalized) Fourier series at infinity:

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{\kappa-1}{2}} e^{2\pi i n z}.$$

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We associate to f an L-function $L(s,f)$ of the Selberg class, defined by the Dirichlet series:

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}}\right).$$

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We will rely crucially on the fact that the Fourier coefficients of f are multiplicative; i.e.,

$$\lambda_f(mn) = \lambda_f(m)\lambda_f(n)$$

for $(m, n) = 1$.

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$$r := \frac{1}{T} \sum_{\gamma_f(i) \leq T} 1 = \frac{1}{\pi} \log \sqrt{q}\, T + O(1).$$

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That is, we are interested in the existence of gaps between zeros much larger than the average spacing.

Previous Results

Similar questions about the spacing between eigenvalues of random matrices are well understood; in that context, it is known that $\Lambda = \infty$.

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By comparison, for the corresponding problem on the Riemann zeta function, the best known result is $\Lambda > 3.18$.

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Theorem (Wirtinger's Inequality)

Let $y : [a, b] \to \mathbb{C}$ be a continuously differentiable function, and suppose that $y(a) = y(b) = 0$. Then

$$\int_a^b |y(x)|^2 \, dx \; \leq \; \left(\frac{b-a}{\pi}\right)^2 \int_a^b |y'(x)|^2 \, dx.$$

Hall's method (cont'd)

Suppose that $y(t)$ is zero when $\frac{1}{2} + it$ is a zero of $L(s, f)$. If we let a, b be the imaginary parts γ_i, γ_{i+1} of successive zeros of our L-function and κ be an upper bound on $\gamma_{i+1} - \gamma_i$, then Wirtinger's inequality implies

$$\kappa^2 \geq \pi^2 \frac{\int_T^{2T} |y(\frac{1}{2} + it)|^2 dt}{\int_T^{2T} |y'(t)|^2 dt} - \mathrm{err}$$

where the error arises from our choice of endpoints.

Hall's method (cont'd)

Suppose that $y(t)$ is zero when $\frac{1}{2}+it$ is a zero of $L(s,f)$. If we let a,b be the imaginary parts γ_i, γ_{i+1} of successive zeros of our L-function and κ be an upper bound on $\gamma_{i+1}-\gamma_i$, then Wirtinger's inequality implies

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where the error arises from our choice of endpoints.

By choosing y appropriately, we therefore obtain a lower bound on the largest gap between zeros of $L(s,f)$. We choose y to be the L-function multiplied by an *amplifier* $M(s)$, essentially a truncated version of the Dirichlet series associated to $L(s,f)$.

Shifted Second Moment

Let $y(t) = L(\frac{1}{2} + it, f)M(\frac{1}{2} + it)$ and let

$$F(s, \alpha, \beta) = L(s + \alpha_1, f)L(\bar{s} + \beta_1, \bar{f})M(s + \alpha_2)\overline{M(\bar{s} + \beta_2)}.$$

Then

$$y(t) = F(\tfrac{1}{2} + it, 0, 0),$$

$$y'(t) = \frac{\partial^4}{\partial\alpha_1\partial\beta_1\partial\alpha_2\partial\beta_2} F(\tfrac{1}{2} + it, 0, 0).$$

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Hence, to obtain estimates on both of the integrals in Hall's method, it suffices to study the shifted moment

$$\int_T^{2T} L(\tfrac{1}{2}+\alpha+it, f)L(\tfrac{1}{2}+\beta-it, \bar{f})M(\tfrac{1}{2}+\alpha_2+it)M(\tfrac{1}{2}+\beta_2-it)dt.$$

Amplifier

Choose $M(s)$ to be a tentatively arbitrary Dirichlet polynomial $\sum_{n\le y}\frac{a(n)}{n^s}$. Let $w(t)$ be a smooth function approximating the characteristic function on $[T,2T]$. Then we have

$$\int_{-\infty}^{\infty} L(\tfrac12+\alpha+it,f)L(\tfrac12+\beta-it,\bar f) \cdot M(\tfrac12+\alpha_2+it)M(\tfrac12+\beta_2-it)w(t)dt$$

$$=\sum_{h,k\le y}\frac{a(h)\overline{a(k)}}{h^{\frac12+\alpha_2}k^{\frac12+\beta_2}} \cdot \int_{-\infty}^{\infty}\left(\frac{h}{k}\right)^{-it} L(\tfrac12+\alpha+it,f)L(\tfrac12+\beta-it,\bar f)w(t)dt.$$

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$$= \sum_{h,k\leq y} \frac{a(h)\overline{a(k)}}{h^{\frac{1}{2}+\alpha_2}k^{\frac{1}{2}+\beta_2}} \cdot \int_{-\infty}^{\infty} \left(\frac{h}{k}\right)^{-it} L(\tfrac{1}{2}+\alpha+it, f)L(\tfrac{1}{2}+\beta-it, \bar{f})w(t)dt.$$

We therefore begin by studying this inner integral $I(h, k)$.

Why Study a Twisted Moment?

- Calculations using conjectural higher give rise to improved bounds in other classes of L-functions.
- We can only calculate the second moment for L-function, by twisting we attempt to “mimic” higher moments.
- We also gain parameters to optimize over when we do numerics at the end

Moment Framework

In our approach we follow Hughes and Young's ('08) work on the twisted fourth moment of the Riemann zeta function.

- Calculate an "approximate functional equation."
- Split the integral into diagonal (non-oscillatory) and off-diagonal (oscillatory) pieces.
- Obtain an asymptotic (in T) for the diagonal.
- Convert the off-diagonal into a shifted convolution sum and estimate its size.

The On and Off Diagonal

Proposition (Approximate Functional Equation)

We have

$$L(\tfrac{1}{2}+\alpha+it,f)L(\tfrac{1}{2}+\beta-it,\bar{f}) = I_{\alpha,\beta,t} + \epsilon_f^2 \Psi_{\alpha,\beta,t} I_{-\beta,-\alpha,t}$$

where

$$I_{\alpha,\beta,t} := \sum_{m,n} \frac{\lambda_f(m)m^{-\alpha}\overline{\lambda_f(n)}n^{-\beta}}{\sqrt{mn}} \left(\frac{m}{n}\right)^{-it} V_{\alpha,\beta,t}\left(\frac{\pi^2 mn}{q}\right).$$

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We split $I(h,k)$ along the diagonal $hm = kn$:

$$\begin{aligned} I^{(1)}(h,k) &= \sum_{hm=kn} \frac{\lambda_f(m)m^{-\alpha}\overline{\lambda_f(n)}n^{-\beta}}{\sqrt{mn}} \times \int_{-\infty}^{\infty} V_{\alpha,\beta,t}\left(\frac{\pi^2 mn}{q}\right) w(t)dt \\ &+ \sum_{r>0} \sum_{hm-kn=r} \frac{\lambda_f(m)m^{-\alpha}\overline{\lambda_f(n)}n^{-\beta}}{\sqrt{mn}} \times \int_{-\infty}^{\infty} \left(\frac{hm}{kn}\right)^{-it} V_{\alpha,\beta,t}\left(\frac{\pi^2 mn}{q}\right) w(t)dt \\ &=: I_D^{(1)}(h,k) + I_O^{(1)}(h,k). \end{aligned}$$

An Euler Product

Note that the Rankin-Selberg square of $L(s, f)$ has Dirichlet series

$$L(s, f \times \bar{f}) = \sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2}{n^s}.$$

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By playing with Euler products, we find

$$\begin{aligned}\mathcal{L}_{\alpha,\beta,h,k}(s, f) &:= \sum_{hm=kn} \frac{\lambda_f(m)\overline{\lambda_f(n)}}{n^{1+\alpha+\beta+s}} \\ &= L(1+\alpha+\beta+s, f \times \bar{f}) B_{\alpha,\beta,h,k}(s, f)\end{aligned}$$

where

$$\begin{aligned}B_{\alpha,\beta,h,k}(s, f) = \prod_{p|h} &\frac{\sum_j \lambda_f(p^j)\overline{\lambda_f(p^{h_p+j})} p^{-j\alpha-(h_p+j)\beta-j(1+s)}}{\sum_j \lambda_f(p^j)\overline{\lambda_f(p^j)} p^{-j\alpha-j\beta-j(1+s)}} \\ &\cdot \prod_{p|k} \frac{\sum_j \lambda_f(p^{k_p+j})\overline{\lambda_f(p^j)} p^{-(k_p+j)\alpha-j\beta-j(1+s)}}{\sum_j \lambda_f(p^j)\overline{\lambda_f(p^j)} p^{-j\alpha-j\beta-j(1+s)}}.\end{aligned}$$

Shifting Across Residues

We have

$$I_D^{(1)}(h,k) = \int_{-\infty}^{\infty} w(t) \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} \times \left(\frac{qt^2}{4\pi^2 hk} \right)^s \mathcal{L}_{\alpha,\beta,h,k}(2s,f)\, ds\, dt + O((qT)^{\epsilon}).$$

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Shifting the vertical line of integration to $\mathrm{Re}\, s = -1/4 + \epsilon$, we encounter a simple pole at $s = 0$. The contribution we pick up from this pole turns out to contain most of the mass of the integral. To be precise, we have

$$I_D^{(1)}(h,k) = \int_{-\infty}^{\infty} w(t) \mathcal{L}_{\alpha,\beta,h,k}(0,f)\, dt + O((qhk)^{-1/4+\epsilon} T^{1/2+\epsilon}).$$

Cleaning Up

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$$\varsigma^*(x,y) := \int_{-\infty}^{\infty} \left(\frac{x}{y}\right)^{-it} \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} \left(\frac{qhk}{\pi^2 xy}\right)^{s} g_{\alpha,\beta,t}(s) w(t)\, ds\, dt$$

so that

$$I_O^{(1)}(h,k) = T \sum\sum_{hm-kn>0} \frac{\lambda_f(m) m^{-\alpha} \overline{\lambda_f(n)} n^{-\beta}}{\sqrt{mn}} \varsigma^*(hm,kn).$$

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We then apply a dyadic partition of unity, so that

$$I_O^{(1)}(h,k) = \frac{T}{\sqrt{MN}} \sum_{M,N} \sum_{0<|r|\leq(*)} \lambda_f(m)\overline{\lambda_f(n)} \varsigma(hm,kn) + O(T^{-\infty})$$

and it turns out that we can take $|r| \leq T^{-1+\epsilon}\sqrt{hkmn}$.

Shifted Convolution Sum

Theorem (Blomer, 2004)

For any $\epsilon > 0$, h, k *and* r *be positive integers and* M, N, P_1, P_2 *be real numbers greater than 1, and* ς *be a smooth function supported on* $[M, 2M] \times [N, 2N]$ *such that* $\|\varsigma^{(ij)}\|_\infty \ll (P_1/N)^i (P_2/M)^j$ *for all* $i, j \geq 0$. *Then*

$$\sum_{hk-mn=r} \lambda_f(m)\overline{\lambda_f(n)}\varsigma(x, y) \ll_{\epsilon,q,\kappa,P_1,P_2} (hM + kN)^{1/2+\epsilon}.$$

Asymptotic

Through arguments similar to those of Hughes and Young ('08), we obtain this asymptotic for $I(h, k)$:

$$I(h,k) = \frac{1}{\sqrt{hk}} \int_{-\infty}^{\infty} w(t) \Bigg(\mathcal{L}_{\alpha,\beta,h,k}(0,f) + \epsilon_f^2 \left(\frac{\sqrt{q}t}{2\pi} \right)^{-2\alpha-2\beta} \mathcal{L}_{-\beta,-\alpha,h,k}(0,f)$$

$$- \frac{2c_f G((\alpha+\beta)/2)}{\alpha+\beta} \left(\frac{qt^2}{4\pi^2 hk} \right)^{-(\alpha+\beta)/2} B_{\alpha,\beta,h,k}(-\alpha-\beta,f)$$

$$+ \frac{2\epsilon_f^2 c_f G((\alpha+\beta)/2}{\alpha+\beta} \left(\frac{qt^2 hk}{4\pi^2} \right)^{-(\alpha+\beta)/2} B_{-\beta,-\alpha,h,k}(\alpha+\beta,f) \Bigg) dt$$

$$+ O((hk)^{3/4+\theta/2+\epsilon} T^{1/2+\theta+\epsilon}),$$

where

$$\mathcal{L}_{\alpha,\beta,h,k}(s,f) = \sum_{hm=kn} \frac{\lambda_f(m)\overline{\lambda_f(n)}}{n^{1+\alpha+\beta+s}}.$$

and

$$B_{\alpha,\beta,h,k} = \prod_{p|h} \frac{\sum_j \lambda_f(p^j)\overline{\lambda_f(p^{h_p+j})} p^{-j\alpha-(h_p+j)\beta-j(1+s)}}{\sum_j \lambda_f(p^j)\overline{\lambda_f(p^j)} p^{-j\alpha-j\beta-j(1+s)}}$$

$$\cdot \prod_{p|k} \frac{\sum_j \lambda_f(p^{k_p+j})\overline{\lambda_f(p^j)} p^{-(k_p+j)\alpha-j\beta-j(1+s)}}{\sum_j \lambda_f(p^j)\overline{\lambda_f(p^j)} p^{-j\alpha-j\beta-j(1+s)}}.$$

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& - \frac{2c_f G((\alpha+\beta)/2)}{\alpha+\beta} \left(\frac{qt^2}{4\pi^2 hk} \right)^{-(\alpha+\beta)/2} B_{\alpha,\beta,h,k}(-\alpha-\beta,f) \\
& + \frac{2\epsilon_f^2 c_f G((\alpha+\beta)/2}{\alpha+\beta} \left(\frac{qt^2 hk}{4\pi^2} \right)^{-(\alpha+\beta)/2} B_{-\beta,-\alpha,h,k}(\alpha+\beta,f) \Bigg) dt \\
& + O((hk)^{3/4+\theta/2+\epsilon} T^{1/2+\theta+\epsilon}).
\end{aligned}
$$

Vanishing of the Fourier coefficients causes our main term to vanish occasionally. We must therefore proceed cautiously in summing our asymptotic over h, k.

Current Progress

We choose the coefficients $a(n)$ of $M(s)$ to be $\lambda_f(n)P[n]$, where P is an arbitrary polynomial and

$$P[x] := P\left(\frac{\log(y/x)}{\log y}\right).$$

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We concentrate on the first two terms in our asymptotic for $I(h,k)$. We have

$$\sum_{h,k\leq y} I_1(h,k) = A(0,0,0,0)L(1+\alpha_1-\beta_1, f\times\bar{f})\int_0^1\int_0^1\sum_{n\leq y}\frac{|\lambda_f(n)|^2(\log y/n)^2}{n^{1+\alpha_2-\beta_2}}$$
$$\left(\frac{y}{n}\right)^{-(\alpha_2-\beta_1)x_1-(\alpha_1-\beta_2)x_2}\cdot P\left(\frac{\log y/n}{\log y}(1-x_1)\right)P\left(\frac{\log y/n}{\log y}(1-x_2)\right)dx_1dx_2+O(\log(T)^5)$$

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We plan to execute the summation over n via Abel summation and an effective Perron formula. After this, there are several largely aesthetic changes to make prior differentiating with respect to the shifts and plugging in to Wirtinger's Inequality to get a bound on the gaps.

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