# A Ramsey Theoretic Approach to Finite Fields and Quaternions

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**Definition (Complete Graph)**

The complete graph on $n$ vertices is an undirected graph with $n$ vertices and a unique edge connecting each vertex.

A classic Ramsey problem is avoiding patterns in 2-colorings of the edges of $K_n$, leading to the Friends and Strangers problem.

# Classical Ramsey Theory

## Friends and Strangers Problem

What is the smallest group of people needed to guarantee $k$ mutual friends or $n$ mutual strangers? Equivalently, what is the smallest $m = R(k, n)$ such that a 2-coloring of the edges of $K_m$ contains a red $K_k$ or blue $K_n$.

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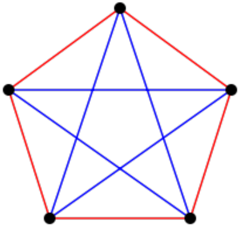
## Friends and Strangers Problem

What is the smallest group of people needed to guarantee $k$ mutual friends or $n$ mutual strangers? Equivalently, what is the smallest $m = R(k, n)$ such that a 2-coloring of the edges of $K_m$ contains a red $K_k$ or blue $K_n$.

$R(3, 3) = 6$. One can prove any coloring of $K_6$ has a red or blue $K_3$, and below is a non-viable coloring of $K_5$.



# Previous Work

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**Rankin (1961)**

Rankin studied integers avoiding 3-term geometric progressions: $b, rb, r^2b$ with $b, rb, r^2b \in \mathbb{Z}$. He used a greedy algorithm to construct a set $G_3^*(\mathbb{Z}) = \{1, 2, 3, 5, 6, 7, 8, 10, 11, 13...\}$.

The elements of $G_3^*(\mathbb{Z})$ can be characterized by their prime exponents.

# Previous Work

## Rankin (1961)

Similarly, one can greedily construct the 3-term arithmetic progression-free set $A_3^*(\mathbb{Z}) = \{0, 1, 3, 4, 9, 10, 12, 13, \ldots\}$.

Write $b, rb, r^2b$ as $p_1^{e_1} \ldots p_n^{e_n}, p_1^{f_1} \ldots p_n^{f_n}, p_1^{g_1} \ldots p_n^{g_n}$. Then since geometric progressions give arithmetic progressions in the prime exponents, $G_3^*(\mathbb{Z})$ is exactly the elements whose prime exponents are in $A_3^*(\mathbb{Z})$.

Calculated the asymptotic density $d(G_3^*(\mathbb{Z})) \approx 0.71974$.

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Calculated the asymptotic density $d(G_3^*(\mathbb{Z})) \approx 0.71974$.

The idea of determining elements in your greedy set by their prime exponents appears a lot in this area.

# Previous Work

**SMALL '14: Generalization to Number Fields**

Studied sets of ideals of number fields' integer rings $\mathcal{O}_K$ that avoid 3-term progressions. Replicated the greedy construction to get large density sets of ideals containing no 3-term progressions, with a similar formula for the density.

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## SMALL '15: Finite Fields and Free Groups

The idea of sets without progressions was extended to finite fields, Hurwitz quaternions, and free groups.

# Preliminaries

## Functon Field

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*We view $\mathbb{F}_q[x]$, with $q = p^n$, as the ring of all polynomials with coefficients in the finite field $\mathbb{F}_q$.*

## Goal

Construct a Greedy Set of polynomials in $\mathbb{F}_q[x]$ free of geometric progressions.

# The Greedy Set

- Rewrite any $f(x)$ as $f(x) = uP_1^{\alpha_1} \cdots P_k^{\alpha_k}$ where $u$ is a unit, and each $P_i$ is a monic irreducible polynomial.
- Exclude $f(x)$ with $\alpha_i \notin A_3^*(\mathbb{Z}) = \{0, 1, 3, 4, 9, 10, 12, 13, \dots\}$.

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**Greedy Set in $\mathbb{F}_q[x]$**

The Greedy Set is exactly the set of all $f(x) \in \mathbb{F}_q[x]$ only with prime exponents in $A_3^*(\mathbb{Z})$

# Asymptotic Density

The *asymptotic density of the greedy set* $G^*_{3,q} \subseteq \mathbb{F}_q[x]$ can be expressed as

$$d(G^*_3) = \left(1 - \frac{1}{q}\right) \prod_{i=1}^{\infty} \prod_{n=1}^{\infty} \left(1 + q^{-n3^i}\right)^{m(n,q)},$$

where $m(n,q) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) q^d$ gives the number of monic irreducibles over $\mathbb{F}_q[x]$.

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where $m(n,q) \;=\; \frac{1}{n}\sum_{d|n} \mu\left(\frac{n}{d}\right) q^d$ gives the number of monic irreducibles over $\mathbb{F}_q[x]$.

Becomes a lower bound when truncated.

# Lower Bound

Table : Lower Bound for Density of $G_3^*(\mathbb{F}_q[x])$.

| $q$ | $d(G_3^*)$ for $\mathbb{F}_q[x]$ |
|---|---|
| 2 | .648361 |
| 3 | .747027 |
| 4 | .799231 |
| 5 | .833069 |
| 7 | .874948 |
| 8 | .888862 |

| $q$ | $d(G_3^*)$ for $\mathbb{F}_q[x]$ |
|---|---|
| 9 | .899985 |
| 25 | .961538 |
| 27 | .964286 |
| 49 | .980000 |
| 125 | .992063 |
| 343 | .997093 |

# Bounds on Upper Densities

We can then use similar combinatorial methods to McNew, Riddell, and Nathanson and O'Byrant to give lower and upper bounds for the upper density of a progression free set for specific values of $q$.

Table : New upper bounds ($q$-smooth) compared to the old upper bounds, as well as the lower bounds for the supremum of upper densities.

| $q$ | New Bound ($q$-smooth) | Old Bound | Lower Bound |
|---|---|---|---|
| 2 | .846435547 | .857142857 | .845397956 |
| 3 | .921933009 | .923076923 | .921857532 |
| 4 | .967684196 | .96774193 | .967680495 |
| 5 | .967684196 | .967741935 | .967680495 |
| 7 | .982448450 | .982456140 | .982447814 |

## Question

Previous work has always been done in a commutative setting. How does non-commutivity affect the problem in, say, free groups or the Hurwitz quaternions $\mathcal{H}$? How does the lack of unique factorization affect the problem in $\mathcal{H}$?

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Previous work has always been done in a commutative setting. How does non-commutivity affect the problem in, say, free groups or the Hurwitz quaternions $\mathcal{H}$? How does the lack of unique factorization affect the problem in $\mathcal{H}$?

Building on methods of McNew, SMALL '14, and Rankin, we construct large subsets of $\mathcal{H}$ that avoid 3-term geometric progressions.

# Types of Quaternions

## Definition

Quaternions constitute the algebra over the reals generated by units $i$, $j$, and $k$ such that

$$i^2 = j^2 = k^2 = ijk = -1.$$

Quaternions can be written as $a + bi + cj + dk$ for $a, b, c, d \in \mathbb{R}$.

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## Definition

The Norm of a quaternion $Q = a + bi + cj + dk$ is given by $\text{Norm}[Q] = a^2 + b^2 + c^2 + d^2$.

# Counting Quaternions

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**Fact**

The number of Hurwitz quaternions of norm $N$ is:

$$S(\{N\}) = 24 \sum_{2 \nmid d | N} d,$$

the sum of the odd divisors of $N$ multiplied by 24.

# Units and Factorization

## Fact

The Hurwitz Order contains 24 units, namely

$$\pm 1, \pm i, \pm j, \pm k \text{ and } \pm \frac{1}{2} \pm \frac{1}{2}i \pm \frac{1}{2}j \pm \frac{1}{2}k.$$

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## Fact

Let $Q$ be a Hurwitz quaternion of norm $q$. For any factorization of $q$ into a product $p_0 p_1 \cdots p_k$ of integer primes, there is a factorization

$$Q = P_0 P_1 \cdots P_k$$

where $P_i$ is a Hurwitz prime of norm $p_i$.

# The Goal

## Goal

Construct and bound Greedy and maximally sized sets of quaternions of the Hurwitz Order free of three-term geometric progressions. For definiteness, we exclude progressions of the form

$$Q, \ QR, \ QR^2$$

where $Q, R \in \mathcal{H}$ and $Norm[R] \neq 1$.

The Hurwitz Quaternions are particularly tricky to work with because they lack unique factorization. While it is hard to characterize a true greedy set by its primes, we can consider the set of Hurwitz Quaternions with norm in $G_3^*(\mathbb{Z})$, which is 3-term progression-free.

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Want: formula for the proportion of quaternions whose norm is divisible by $p^n$ and not $p^{n+1}$. We study the proportion of (Hurwitz) quaternions up to norm $N$ whose norm is exactly divisible by $p^n$.

$$\frac{\text{Quats with norm div by } p^n \text{ - Quats with norm div by } p^{n+1}}{\text{Quats with norm} \leq N}$$

$$= \frac{(\text{Quats with norm } p^n)(\text{Quats with norm} \leq N/p^n)}{24 \cdot (\text{Quats with norm} \leq N)} - \frac{(\text{Quats with norm } p^{n+1})(\text{Quats with norm} \leq N/p^{n+1})}{24 \cdot (\text{Quats with norm} \leq N)}$$

$$= \frac{\left(\sum_{2\nmid d|p^n} d\right) V_4(\sqrt{N/p^n}) - \left(\sum_{2\nmid d|p^{n+1}} d\right) V_4(\sqrt{N/p^{n+1}})}{V_4(\sqrt{N})} + \text{error},$$

where $V_4(M)$ denotes the volume of a 4-dimensional sphere of radius $M$. For $p$ odd

$$\sum_{2\nmid d|p^n} d = 1 + \cdots + p^n = (p^{n+1} - 1)/(p - 1).$$

For $p = 2$, the quantity is 1.

We sum up probabilities of having norm divisible by $p^n$ to find the proportion of quaternions whose norm is exactly divisible by $p^n$ for $p$ fixed, $n \in A_3^*(\mathbb{Z})$:

$$\sum_{n \in A_3^*(\mathbb{Z})} \frac{p^{n+3} - p^{n+2} - p^2 + 1}{p^2(p-1)p^{2n}}.$$

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To find the density of $\{q \in \mathcal{H} : \text{Norm}[q] \in G_3^*(\mathbb{Z})\}$, we multiply these terms to get all norms with prime powers in $A_3^*(\mathbb{Z})$, i.e., norms in $G_3^*(\mathbb{Z})$.

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$$\begin{aligned} d(\{q \in \mathcal{H} : \text{N}[q] \in G_3^*(\mathbb{Z})\}) &= \left[\sum_{n \in A_3^*(\mathbb{Z})} \frac{2^2 - 1}{2^2 2^{2n}}\right] \\ &\quad \times \prod_{p \text{ odd}} \left[\sum_{n \in A_3^*(\mathbb{Z})} \frac{p^{n+3} - p^{n+2} - p^2 + 1}{p^2(p-1)p^{2n}}\right] \\ &\approx .77132. \end{aligned}$$

Instead of studying large density sets avoiding 3-term progressions, we can also try to maximize the upper density.

## Definition (Upper Density)

The upper density of a set $A \subset \mathcal{H}$ is

$$\limsup_{N\to\infty} \frac{|A \cap \{q \in \mathcal{H} : \text{Norm}[q] \leq N\}|}{|\{q \in \mathcal{H} : \text{Norm}[q] \leq N\}|}$$

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We wish to study lower bounds for the supremum of upper densities of 3-term progression-free sets.

# Lower Bound for the Supremum

For a lower bound, we construct a set with large upper density. Consider

$S_N =$ $\left(\frac{N}{4}, N\right]$

Then the quaternions with norm in $S_N$ have no 3-term progressions in their norms, and thus no 3-term progressions in the elements themselves.

By spacing out copies of $\{q \in \mathcal{H} : \text{Norm}[q] \in S_N\}$, we construct a set with upper density

$$\lim_{N \to \infty} \frac{\{q \in \mathcal{H} : \text{Norm}[q] \in S_N\}}{\{q \in \mathcal{H} : \text{Norm}[q] \leq N\}} \approx .946589.$$

# Lower Bound for the Supremum

For a lower bound, we construct a set with large upper density.
Consider

$$S_N = \left(\frac{N}{9}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$$

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# Lower Bound for the Supremum

For a lower bound, we construct a set with large upper density. Consider

$$S_N = \left(\tfrac{N}{24}, \tfrac{N}{12}\right] \cup \left(\tfrac{N}{9}, \tfrac{N}{8}\right] \cup \left(\tfrac{N}{4}, N\right]$$

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For a lower bound, we construct a set with large upper density. Consider

$$S_N = \left(\frac{N}{32}, \frac{N}{27}\right] \cup \left(\frac{N}{24}, \frac{N}{12}\right] \cup \left(\frac{N}{9}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$$

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For a lower bound, we construct a set with large upper density. Consider

$$S_N = \left(\frac{N}{40}, \frac{N}{36}\right] \cup \left(\frac{N}{32}, \frac{N}{27}\right] \cup \left(\frac{N}{24}, \frac{N}{12}\right] \cup \left(\frac{N}{9}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$$

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$S_N = \left(\frac{N}{48}, \frac{N}{45}\right] \cup \left(\frac{N}{40}, \frac{N}{36}\right] \cup \left(\frac{N}{32}, \frac{N}{27}\right] \cup \left(\frac{N}{24}, \frac{N}{12}\right] \cup \left(\frac{N}{9}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$

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# The Greedy Set

Recall Rankin's greedy set, $G_3^*$:
$1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21...$

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Norms of elements in our greedy set:
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# The Greedy Set

Recall Rankin's greedy set, $G_3^*$:
$1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21...48, 51...$

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Reasons for discrepancies:

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Norms of elements in our greedy set:
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Reasons for discrepancies: Try $31^2$. Recall

$$S(\{N\}) = 24 \sum_{2 \nmid d | N} d.$$

Then $S(\{31^2\}) = 24 \sum_{2 \nmid d | 31^2} d$. However, the number of ways to write a quaternion of norm $31^2$ as the square of a quaternion of norm 31 multiplied by a unit is

$$S(\{31^2\}) \geq 24 \sum_{2 \nmid 31 d | 31^2} d = 24 * 31 \sum_{2 \nmid d | 31} d > 24 S(\{31\}).$$

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# Future Work

Questions

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- How does the density of the set constructed by taking norms from Rankin's greedy set of integers compare to the density of the greedy subset of quaternions?

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- How does the density of the set constructed by taking norms from Rankin's greedy set of integers compare to the density of the greedy subset of quaternions?
- Can a trivial upper or lower bound be found for the greedy set?

# Upper Bound for the Supremum

From a progression-free set, we must exclude one of $\{b, br, br^2\}$ from any such tuple. By looking at a large number of disjoint such tuples, we can force a proportion of exclusions. Picking $r$ minimal will yield more exclusions.

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Fix $r$ of norm 2, consider quaternions up to norm $M$. If $\text{Norm}[b] \leq M/4$ and $N[b]$ has no factor of $r$, then $b, br, br^2$ forms a progression, and different $b$ yield disjoint tuples.

As a result, we can exclude

$$\lim_{M\to\infty} 1-(\text{Proportion coprime to } r)\frac{\text{Number of quats up to } M/4}{\text{Number of quats up to } M}$$
$$= 1-(3/4)(1/2^4)$$
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$$1 - \frac{3}{2^6} \cdot \sum_{i=0}^{\infty} \frac{1}{2^{6i}} \approx .952381.$$