Classification of Crescent Configurations on 4 and 5 points

Rebecca F. Durst, Max Hlavacek, Chi Huynh

Advisors: Steven J. Miller and Eyvindur A. Palsson

Number Theory and Harmonic Analysis Group - SMALL 2016 - Williams College

rfd1@williams.edu; mhlavacek@hmc.edu; nhuynh30@gatech.edu

Abstract

We take a new approach to investigating crescent configurations using techniques from distance geometry and graph theory that have allowed us to provide a method for classifying all configurations on n points up to graph isomorphism. Furthermore, we have definitively proven that there exist only three possible realizations for a configuration on four points and have decreased the number of configurations on five points from 12,600 candidates to no more than 26 potentially realizable final configurations. We then return to Erdős' original question regarding the existence of these configurations with a new approach using distance geometry that has proven to be an effective method for turning previously intractable problems into a more solvable form.

1. Overview

Definition 1.1. ***General Position** in $\mathbb{R}^d$: No d+1 points on the same hyperplane and no d+2 points on the same hypersphere.*

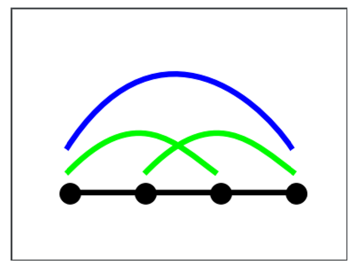


Figure 1: *Non-example of general position*

Definition 1.2. *We say n points are in crescent configuration (in $\mathbb{R}^d$) if they lie in general position in $\mathbb{R}^d$ and determine $n-1$ distinct distances, such that for every $1 \le i \le n-1$ there is a distance that occurs exactly i times.*

Why do we care about Crescent Configurations?

◇ **Erdős Conjecture (1989)**: There exists an N sufficiently large such that no crescent configuration exists on N points.

◇ **Pomerance and Palásti (1989)**: Construction for n=5, n=6 n=7,n=8.

◇ **SMALL 2015**: There exists a crescent configurations on d points in $\mathcal{R}^{d-2}$

Figure 2: *Crescent Configurations on 5,6,7,8 points*

Issues with the Construction of Crescent Configurations

◇Mostly guess and check

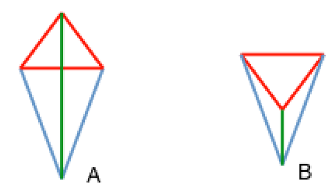
◇Difficult to combinatorially demonstrate the conditions of general positions and geometric realizability.

2. Counting Distinct Crescent Configurations

2.1 Isomorphism of Crescent Configurations

- **Distance Coordinate:** The distance coordinate, D_a of a point a is the set of all distances, counting multiplicity, between a and the other points in a set, $\mathcal{P}$.
- **Distance Set:** The distance set, $\mathcal{D}$, corresponding to a set of points, $\mathcal{P}$, is the set of the distance coordinates for each point in the $\mathcal{P}$.

Theorem 2.1 (Durst-Hlavacek-Huynh 2016). *Let A and B be two crescent configurations on the same number of points n. If A and B have the same distance sets, then there exists a graph isomorphism $A \to B$.*



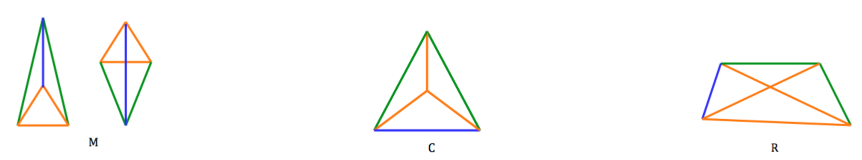
$$\begin{pmatrix} 0 & d_3 & d_1 & d_3 \\ d_3 & 0 & d_2 & d_3 \\ d_1 & d_2 & 0 & d_2 \\ d_3 & d_3 & d_2 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & d_3 & d_3 & d_2 \\ d_3 & 0 & d_3 & d_1 \\ d_3 & d_3 & 0 & d_2 \\ d_2 & d_1 & d_2 & 0 \end{pmatrix}$$

Figure 3: *Two Isomorphic Crescent Configurations on 4 points*

2.2 Result

Theorem 2.2 (Durst-Hlavacek-Huynh 2016). *Given a set of three distinct distances, $\{d_1, d_2, d_3\}$, on four points in crescent configuration, there are only three allowable crescent configurations up to graph isomorphism*

We label these M-type, C-type, and R-type, respectively.



Theorem 2.3 (Durst-Hlavacek-Huynh 2016). *Given a set of four distinct distances, $\{d_1, d_2, d_3, d_4\}$, on five points in crescent configuration, there are 27 allowable crescent configurations up to graph isomorphism.*

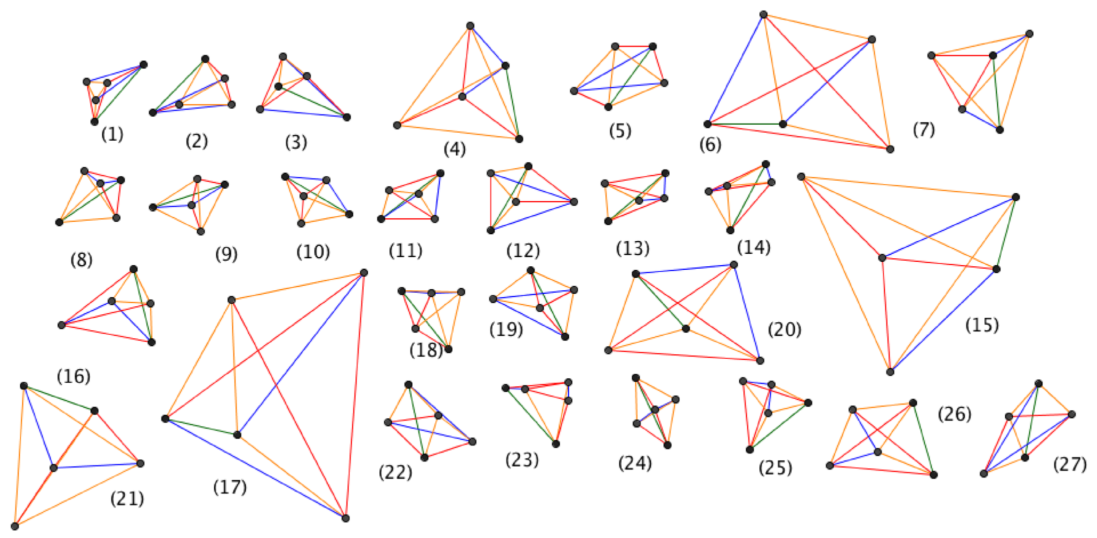


Figure 5: *27 crescent configurations on five points.*

3. Geometric Realizability

Question: Given a distance set $\mathcal{D}$, can we find a set of points in a crescent configuration with $\mathcal{D}$ as its distance set in $\mathbb{R}^n$?

Main tool - Cayley Menger Matrix: The Cayley Menger matrix for a set of n points $\{P_1, P_2, \dots P_n\}$ is an $(n+1) \times (n+1)$ matrix of the following form:

$$\begin{pmatrix} 0 & d_{1,2}^2 & \dots & d_{1,n}^2 & 1 \\ d_{2,1}^2 & 0 & \dots & d_{2,n}^2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{n,1}^2 & d_{n,2}^2 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}$$

where $d_{i,j}$ is the distance between P_i and P_j.

Theorem 3.1 (Sommerville 1958). *A distance set corresponding to 4 points is geometrically realizable in $\mathbb{R}^2$ if and only if the Cayley-Menger matrix is not invertible.*

Solutions for a Given Crescent Configuration Type

We can fix one of the unknown distances and use Cayley-Menger determinants to find a system of equations that yields geometrically realizable distances.

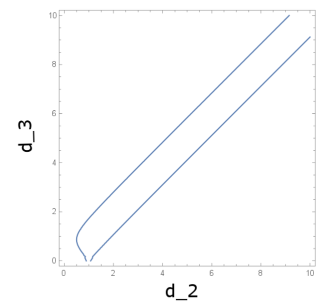


Figure 6: *Possible values for d_2, d_3 for the M-type when $d_1 = 1$*

4. Rigidity of Crescent Configurations - An Independent Validation

Question: Given $n-1$ distinct distances with prescribed multiplicities, can we realize two different crescent configurations on n points?

4.1 Preliminaries

- Let $G = (V, E)$ be a graph with some pairwise associated distance measurements. A realization f of G is a function that maps the vertices of G to coordinates in some Euclidean space such that the distance measurements are realized. $f(G)$ is called a framework.
- $f(G)$ is **flexible** if and only if it can be continuously deformed while preserving the distance constraints; otherwise it is **rigid**. $f(G)$ is **redundantly rigid** if and only if one can remove any edge and the remaining framework is rigid.

Theorem 4.1 (Hendrickson 1992). *A framework $f(G)$ is rigid if and only if its rigidity matrix has rank exactly equal to $S(n,d)$, the number of allowed motions, which equals $nd - d(d+1)/2$ for $n \ge d$ and $n(n-1)/2$ otherwise.*

4.2 Results for n = 4

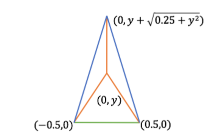


Figure 7: *Realization obtained by fixing $d_1 = 1$*

Type C defines a rigid graph

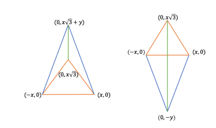


Figure 8: *Two realizations of type M: M_1 and M_2*

Type M defines a rigid graph

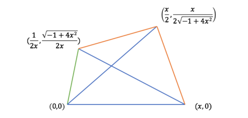


Figure 9: *Realization obtained by fixing $d_1 = 1$*

Type R defines a redundantly rigid graph

5. Acknowledgements

- Williams College Finnerty Fund
- Williams College and SMALL REU
- NSF Grants DMS1265673, DMS1561945 and DMS1347804
- Prof. A.P.W.B. Dumbledore and Prof. M. McGonagall for their undying support.

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