Random Matrix Theory: Checkerboard Matrices and Limiting Spectral Distributions

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Introduction

Background An important problem in random matrix theory involves investigating the distribution of eigenvalues of random matrix ensembles. Such a study has broad applications ranging diverse fields in pure mathematics, physics, and economics.

Wigner's Semi-circle Law In the ensemble of $N \times N$ real Wigner matrices, for almost all matrices A as N approaches infinity

$$\mu_{A,N} \to \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases} \tag{1}$$

where $\mu_{A,N}$ is the probability measure

$$\mu_{A,N} = \frac{1}{N}\sum_{i=1}^{N} \delta\left(x - \frac{\lambda_i}{2\sqrt{N}}\right) \tag{2}$$

for $\{\lambda_i\}_{i=1}^N$ the eigenvalues of A.

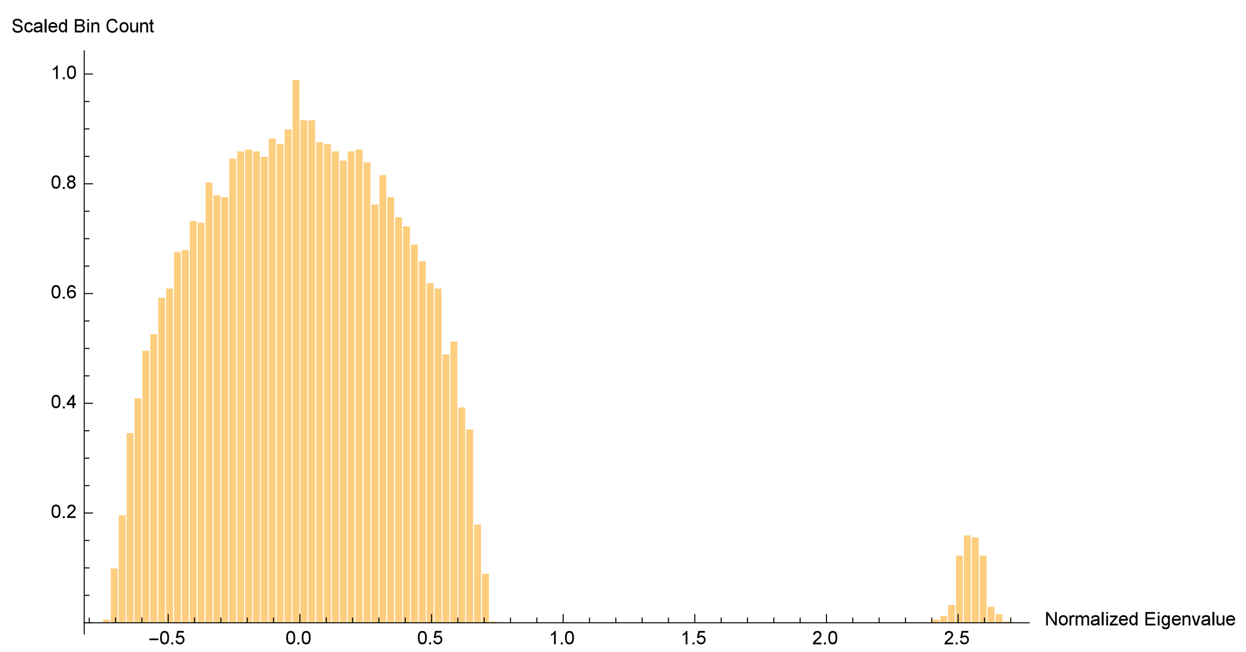
The k-Checkerboard Matrices: We examine an ensemble of matrices which exhibit atypical limiting behavior and require novel techniques to analyze. Namely, k-checkerboard matrices are a natural generalization to matrices of the form:

$$A = \begin{bmatrix} w & a_{0\,1} & w & a_{0\,3} & w & \cdots & a_{0\,N-1} \\ a_{0\,1} & w & a_{1\,2} & w & a_{1\,4} & \cdots & w \\ w & a_{1\,2} & w & a_{2\,3} & w & \cdots & a_{2\,N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{0\,N-1} & w & a_{2\,N-1} & w & a_{4\,N-1} & \cdots & w \end{bmatrix} \tag{3}$$

More precisely, given a fixed value w (we take $w = 1$) and $k \in \mathbb{N}$ with $k \mid N$, then an $N \times N$ matrix A is a k-checkerboard matrix if $A = (a_{ij})$ is given by

$$a_{ij} = \begin{cases} \mathcal{N}(0,1) & \text{if } i \not\equiv j \mod k \\ w & \text{if } i \equiv j \mod k. \end{cases} \tag{4}$$

Abnormal Behavior



Some Heuristics

- The mean of the blip is roughly N/k.
- The variance of the blip is independent of N.

Methods

Markov's Method of Moments We attempt to show a typical eigenvalue measure $\mu_{A,N}(x)$ converges to a probability distribution P by controlling convergence of average moments of the measures as $N \to \infty$ to the moments of P.

In order to calculate the moments of the eigenvalue distribution, we use the:

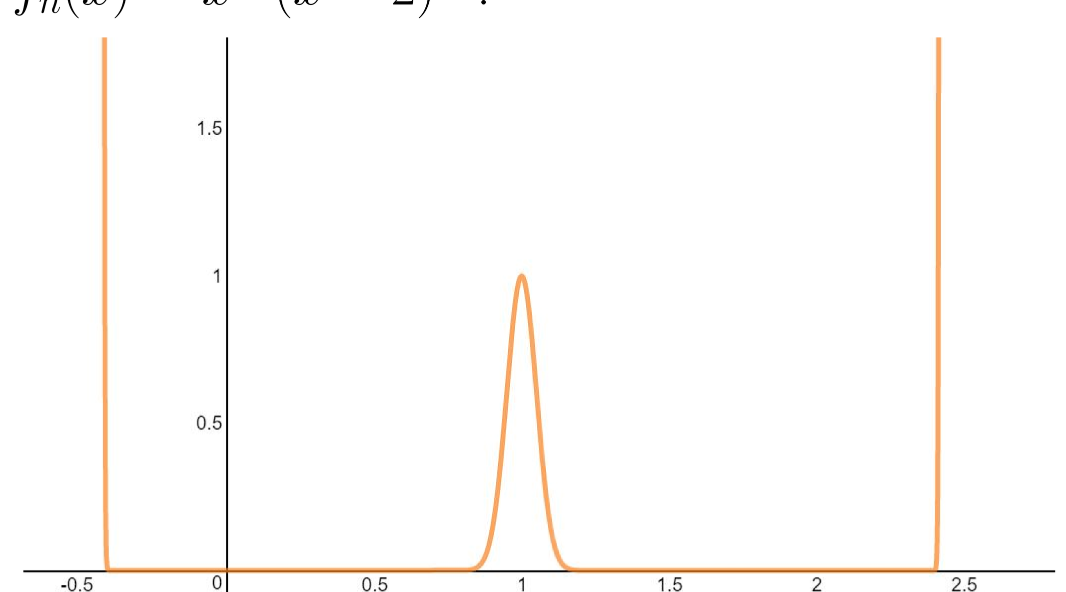
Eigenvalue Trace Lemma For any non-negative integer k, if A is an $N \times N$ matrix with eigenvalues $\lambda_i(A)$, then

$$\text{Trace}\left(A^k\right) = \sum_{i=1}^{N} \lambda_i(A)^k.$$

We construct a spectral measure

$$\mu_{A,N} := \sum_{\lambda_i} f_n\left(\frac{k\lambda}{N}\right)\delta\left(x - \left(\lambda - \frac{N}{k}\right)\right) \tag{5}$$

for which $f_n(x) = x^{2n}(x-2)^{2n}$.



We get the following formula for the average m^{th} moment, $M_k(N) = \mathbb{E}[M_k(A_N)]$, is:

$$\mu_m = \left(\frac{k}{N}\right)^{2n}\sum_{j=0}^{2n}\binom{2n}{j}\sum_{i=0}^{m+j}\binom{m+j}{i}\left(-\frac{N}{k}\right)^{m-i}\mathbb{E}\ \text{Tr}\ A^{2n+i} \tag{6}$$

Trace Computation and Graph Walks We compute the trace term in the Equation **??** via the identity

$$\mathbb{E}\ \text{Tr}\ A^m = \sum_{i_1,\ldots,i_m=1}^{N} a_{i_1 i_2}\cdots a_{i_{m-1}i_m}a_{i_m i_1}. \tag{7}$$

We have now reduced to a combinatorial problem of determining the main order term in the contribution of the cyclic products. We broke the problem further down into choosing length of the cyclic product, number of blocks of a's and their locations, choosing the congruence classes of indices and finally the contribution from choosing the indices themselves.

Main Results

The **Zero-Diagonal Gaussian Orthogonal Ensemble** is defined as follows: A symmetric matrix $A \in \mathbb{M}_N(\mathbb{R})$ is in the zero-diagonal Gaussian Orthogonal Ensemble if $A = (a_{ij})$ is given by

$$a_{ij} = \begin{cases} \mathcal{N}(0,1) & \text{if } i \ne j \\ 0 & \text{if } i = j. \end{cases} \tag{8}$$

$$A = \begin{bmatrix} 0 & a_{0\,1} & a_{0\,2} & a_{0\,3} & a_{0\,4} & \cdots & a_{0\,N-1} \\ a_{0\,1} & 0 & a_{1\,2} & a_{1\,3} & a_{1\,4} & \cdots & a_{1\,N-1} \\ a_{0\,2} & a_{1\,2} & 0 & a_{2\,3} & a_{2\,4} & \cdots & a_{2\,N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{0\,N-1} & a_{1\,N-1} & a_{2\,N-1} & a_{3\,N-1} & a_{4\,N-1} & \cdots & 0 \end{bmatrix} \tag{9}$$

We write the expectation over the $k \times k$ zero-diagonal GOE as $\mathbb{E}_k$.

Theorem 1

As $N \to \infty$, the expectation of the centered moments μ'_m of the k-checkerboard matrix over $\mathbb{R}$ converges to

$$\mu'_m = \mathbb{E}_k\ \text{Tr}\ A^m. \tag{10}$$

We proved analogous theorems for complex and quaternion k-checkerboard Hermitian matrices.

Furthermore, for the sequence of averaged measures μ_N over these matrices, using the Borel-Cantelli Lemma, we are able to also show that

Theorem 2

As $N \to \infty$, the averaged measures μ_N over the k-checkerboard matrices over $\mathbb{R}$ converges weakly almost surely to the standard spectral measure of the $k \times k$ zero-diagonal GOE.

This proof also generalizes to the complex and quaternion cases.

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