Second Moments of Fourier Coefficients of Convolutions of Families of L-functions

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1. Introducing Second Moments of Fourier Coefficients

1.1 Background

A key property of L-functions is the existence of an Euler product, from which we may derive results analogous to the explicit formula for the Riemann Zeta function, as well as other deep properties of objects that L-functions encode information about. In general, the L-function can be represented both as a series and the Euler product

$$L(s,\pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s} = \prod_p \prod_{i=1}^{n} (1-\alpha_{\pi,i}p^{-s})^{-1},$$

where the outer product is taken over primes p. We refer to the $\alpha_{\pi,i}$ as the Satake parameters.

In general, we define the *local second moment of the Fourier coefficients of a family $\mathcal{F}$ of L-functions* via

$$M_2(\mathcal{F},p) = \sum_{\pi\in\mathcal{F}} \lambda_\pi(p)^2,$$

where the sum is over all L-functions in the family. (Sometimes this is slightly modified as the family becomes larger and more parameters control its size.)

Then, we define the *second moment of the Fourier coefficients of a family $\mathcal{F}$ of L-functions* in terms of the local second moment via

$$M_2(\mathcal{F},X) = \sum_{p<X} M_2(\mathcal{F},p).$$

1.2 Tools

Previous families of L-functions that have been studied include Dirichlet L-functions, symmetric lifts of cuspidal newforms, and L-functions of elliptic curves. There several essential averaging formulas that enable our calculations.

Theorem 1 (Orthogonality relation for Dirichlet Characters): Let $\mathcal{D}_q$ denote the family of Dirichlet characters of prime modulus q. Then,

$$\sum_{\chi\in\mathcal{D}_q} \chi(p)^2 = \begin{cases} q-2 & \text{if } p\equiv\pm1 \pmod q \\ -1 & \text{if } p\not\equiv\pm1 \pmod q. \end{cases}$$

Theorem 2 (Petersson Formula): For square-free level q and even k let $H^*_{k,q}(\chi_0)$ be the space of cuspidal newforms of level q, weight k, and trivial center character. Furthermore, if n is a square such that $(n,q^2)\mid q$,

$$\sum_{f\in H^*_{k,q}(\chi_0)} \lambda_f(n) = \frac{k-1}{12}\frac{\varphi(q)}{\sqrt{n}} + O\left((n,q)^{-\frac{1}{2}} n^{\frac{1}{6}} k^{\frac{2}{3}} q^{\frac{2}{3}}\right)$$

where the main term exists if and only if $n^{\frac{9}{7}} \leq k^{\frac{16}{21}} q^{\frac{6}{7}}$.

Theorem 3 (Birch): If $\mathcal{F}$ is the family of all elliptic curves of the form

$$\mathcal{E}: y^2 = x^3+ax+b, a,b\in\mathbb{Z},$$

then

$$M_2(\mathcal{F},p) = \sum_{\mathcal{E}\in\mathcal{F}} a_{\mathcal{E}}(p)^2 = p^3-p^2.$$

2. Previous Work

The following table demonstrates calculations previously done by Miller and his colleagues.

Family	Second Moment
$\mathcal{D}_q$	$\frac{X}{\log X} + \frac{X}{\log^2 X} + O\left(\frac{X}{\log^3 X}\right)$
$H^*_{k,q}(\chi_0), q\to\infty$	$\frac{1}{48}\frac{X^{2\delta+\varepsilon}}{\log X^\varepsilon} + \frac{1}{48}\frac{X^{2\delta+\varepsilon}}{\log^2 X^\varepsilon} + O\left(\frac{X^{2\delta+\varepsilon}}{\log^3 X^\varepsilon}\right)$
$H^*_{k,q}(\chi_0)$	$\frac{X^{2\delta_1+2\delta_2+\varepsilon}}{96\log X^{\delta_2\log X^\varepsilon}} + \frac{X^{2\delta_1+2\delta_2+\varepsilon}}{96\log X^{\delta_2}\log^2 X^\varepsilon} - \frac{X^{2\delta_1+2\delta_2+\varepsilon}}{192\log^2 X^{\delta_2}\log X^\varepsilon} + O\left(\frac{X^{2\delta_1+2\delta_2+\varepsilon}}{\log^2 X^{\delta_2}\log^2 X^\varepsilon}\right)$

Remark: the $\delta_1,\delta_2,\varepsilon$ parameters control the size of the family $H^*_{k,q}(\chi_0)$ that we take with respect to X, revealing different asymptotic behaviors. In particular, ε controls the number of primes averaged over, δ_1 controls the size of the weight k, and δ_2 controls the size of the level q. $q\to\infty$ indicates that we do not average over q but instead take the limit.

3. The Rankin-Selberg Convolution and New Results

3.1 Rankin-Selberg Convolution

Directly related to the Rankin-Selberg method for integral representations of L-functions, the Rankin-Selberg convolution for two families of L-functions generates a new family of L-functions via pairwise multiplication of the Satake parameters. If $L(s,\pi_1), L(s,\pi_2)$ have Satake parameters $\{\alpha_{\pi_1,i}(p)\}_{i=1}^n, \{\alpha_{\pi_2,j}(p)\}_{j=1}^m$, then the convolution has Satake parameters

$$\{\alpha_{\pi_1\times\pi_2,k}(p)\}_{k=1}^{nm} = \{\alpha_{\pi_1,i}(p)\alpha_{\pi_2,j}(p)\}_{\substack{1\leq i\leq n\\ 1\leq j\leq m}}.$$

The arithmetic properties of the convolution for several families can provide insight into the nature of the constituent families.

3.2 Convolutions with families of Dirichlet L-functions

Convolved Families	Second Moment
$\mathcal{D}_{q_1}\times\mathcal{D}_{q_2}$	$\frac{X}{\log X} + \frac{X}{\log^2 X} + O\left(\frac{X}{\log^3 X}\right)$
$\mathcal{D}_q\times H^*_{k,q}(\chi_0), q\to\infty$	$\frac{1}{48}\frac{X^{2\delta+\varepsilon}}{\log X^\varepsilon} + \frac{1}{48}\frac{X^{2\delta+\varepsilon}}{\log^2 X^\varepsilon} + O\left(\frac{X^{2\delta+\varepsilon}}{\log^3 X^\varepsilon}\right)$
$\mathcal{D}_q\times H^*_{k,q}(\chi_0)$	$\frac{X^{2\delta_1+2\delta_2+\varepsilon}}{96\log X^{\delta_2\log X^\varepsilon}} + \frac{X^{2\delta_1+2\delta_2+\varepsilon}}{96\log X^{\delta_2}\log^2 X^\varepsilon} - \frac{X^{2\delta_1+2\delta_2+\varepsilon}}{192\log^2 X^{\delta_2}\log X^\varepsilon} + O\left(\frac{X^{2\delta_1+2\delta_2+\varepsilon}}{\log^2 X^{\delta_2}\log^2 X^\varepsilon}\right)$

Note: the above moments are all identical to the table of previous calculations.

3.3 Convolutions with families of symmetric lifts of Cuspidal Newforms

Convolved Families	Second Moment
$H^*_{k_1,q_1}(\chi_0)\times H^*_{k_2,q_2}(\chi_0), q_1,q_2\to\infty$	$\frac{X^{2\delta_1+2\delta_2+\varepsilon}}{2304\log X^\varepsilon} + \frac{X^{2\delta_1+2\delta_2+\varepsilon}}{2304\log^2 X^\varepsilon} + O\left(\frac{X^{2\delta_1+2\delta_2+\varepsilon}}{\log^3 X^\varepsilon}\right)$
$H^*_{k_1,q_1}(\chi_0)\times H^*_{k_2,q_2}(\chi_0)$	$\begin{cases} \frac{X^{2\delta_1+2\delta_2+2\rho_1+2\rho_2+\varepsilon}}{9216\log X^\varepsilon\log X^{\rho_1}\log X^{\rho_2}}\left(1+\frac{1}{\log X^\varepsilon}\right) & \text{if } \varepsilon<\rho_1,\rho_2 \\ \frac{X^{2\delta_1+2\delta_2+2\rho_1+2\rho_2+\varepsilon}}{9216\log X^\varepsilon\log X^{\rho_1}\log X^{\rho_2}}\left(1-\frac{1}{2\log X^{\rho_1}}\right) & \text{if } \rho_1<\varepsilon,\rho_2 \\ \frac{X^{2\delta_1+2\delta_2+2\rho_1+2\rho_2+\varepsilon}}{9216\log X^\varepsilon\log X^{\rho_1}\log X^{\rho_2}}\left(1-\frac{1}{2\log X^{\rho_2}}\right) & \text{if } \rho_2<\varepsilon,\rho_1 \\ \frac{X^{2\delta_1+2\delta_2+2\rho_1+2\rho_2+\varepsilon}}{9216\log X^\varepsilon\log X^{\rho_1}\log X^{\rho_2}}\left(1+\frac{1}{2\log X^\varepsilon}\right) & \text{if } \varepsilon=\rho_1<\rho_2 \\ \frac{X^{2\delta_1+2\delta_2+2\rho_1+2\rho_2+\varepsilon}}{9216\log X^\varepsilon\log X^{\rho_1}\log X^{\rho_2}}\left(1+\frac{1}{2\log X^\varepsilon}\right) & \text{if } \varepsilon=\rho_2<\rho_1 \\ \frac{X^{2\delta_1+2\delta_2+2\rho_1+2\rho_2+\varepsilon}}{9216\log X^\varepsilon\log X^{\rho_1}\log X^{\rho_2}}\left(1-\frac{1}{\log X^{\rho_1}}\right) & \text{if } \rho_1=\rho_2<\varepsilon \\ \frac{X^{2\delta_1+2\delta_2+2\rho_1+2\rho_2+\varepsilon}}{9216\log X^\varepsilon\log X^{\rho_1}\log X^{\rho_2}}\left(1-\frac{3}{16\log^2 X^\varepsilon}\right) & \text{if } \varepsilon=\rho_1=\rho_2. \end{cases}$

3.4 Convolutions with families of Elliptic Curve L-functions

Below, we denote the family of all elliptic curves (as in Theorem 3) by $\mathcal{F}_{\mathcal{E}}$.

Convolved Families	Second Moment
$\mathcal{D}_q\times\mathcal{F}_{\mathcal{E}}$	$\frac{1}{4}\frac{X^4}{\log X} + \frac{1}{16}\frac{X^4}{\log^2 X} + O\left(\frac{X^4}{\log^3 X}\right)$
$\mathcal{F}_{\mathcal{E}}\times\mathcal{F}_{\mathcal{E}}$	$\frac{1}{7}\frac{X^7}{\log X} + \frac{1}{49}\frac{X^7}{\log^2 X} + O\left(\frac{X^7}{\log^3 X}\right)$

4. References

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