Linear Recurrence Relations with Non-constant Coefficients and Benford's Law

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Motivating Question: For a nice data set, such as the Fibonacci numbers, stock prices, street addresses of college employees and students, ..., what percent of the leading digits are 1?

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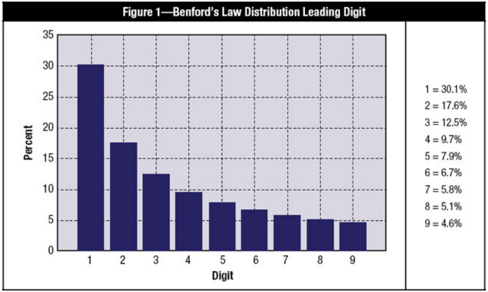
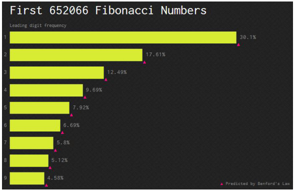


Answer: Benford's law!

Examples with First Digit Bias

Fibonacci numbers



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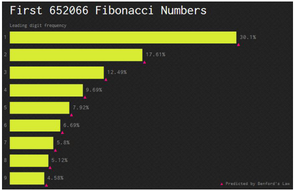
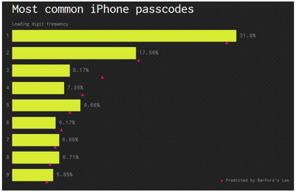


Most common iPhone passcodes



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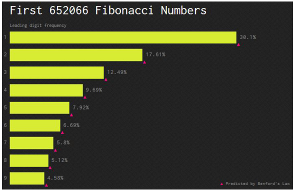

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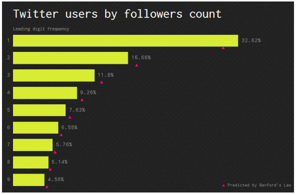

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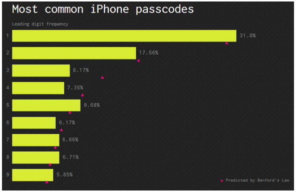

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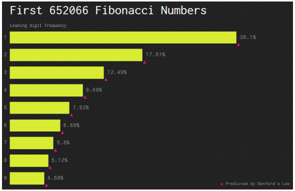

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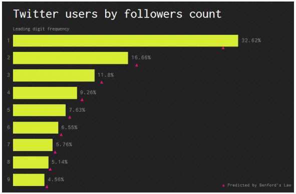

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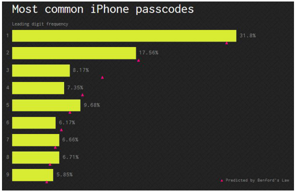
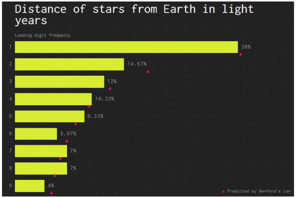


Distance of stars from Earth



Applications

- Analyzing round-off errors.
- Determining the optimal way to store numbers.
- Detecting tax and image fraud, and data integrity.

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- Example: The Fibonacci Sequence 1, 2, 3, 5, 8, 13, 21, 34, ...
- Question: Non-constant coefficients?

Outline

- Benford's Law.
- Linear Recurrence relations (degree 2 and higher degree).
- Multiplicative Recurrence relations.
- Open problems and references.

Equidistribution and Benford's Law

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Strong Benford

A set of numbers is Strong Benford (base b) if the probability of observing a significand in $[1, s)$ is $log_b(s)$. Then the probability of observing a significand in $[s, s+1)$ is $\log_b\left(1+\frac{1}{s}\right)$.

Equidistribution and Benford's Law

Equidistribution

$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a, b]$ tends to $b - a$:

$$\frac{\#\{n \leq N : y_n \bmod 1 \in [a, b]\}}{N} \rightarrow b - a.$$

Theorem

$\beta \notin \mathbb{Q}$, $n\beta$ is equidistributed mod 1.

Logarithms and Benford's Law

Fundamental Equivalence

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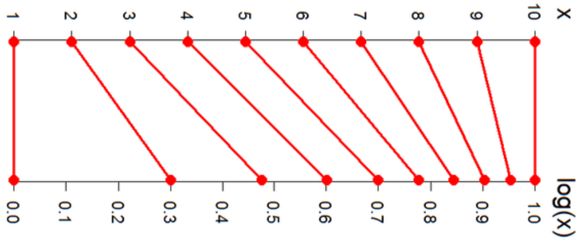
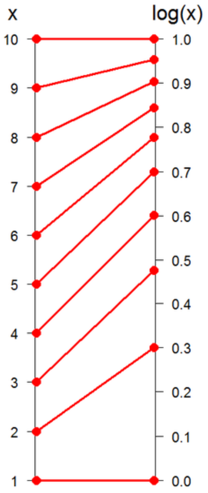


Logarithms and Benford's Law

$$\begin{aligned}&\mathrm{Prob}(\text{leading digit } d)\\ &= \log_{10}(d+1) - \log_{10}(d)\\ &= \log_{10}\left(\frac{d+1}{d}\right)\\ &= \log_{10}\left(1+\frac{1}{d}\right).\end{aligned}$$

Have Benford's law $\leftrightarrow$ mantissa (fractional part) of logarithms of data are uniformly distributed



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- Remember: $\beta \notin \mathbb{Q}$, $n\beta$ is equidistributed mod 1.
- 2^n is Benford base 10 as $\log_{10} 2 \notin \mathbb{Q}$.
- Fibonacci numbers are Benford base 10.
  Binet: $a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$.

Useful Theorem

It suffices to analyze the main term of the sequence:

Theorem

If a sequence $\{a_n\}$ *is Benford and* $\lim_{n\to\infty} b_n = a_n$ *then* $\{b_n\}$ *is Benford as well.*

Linear Recurrence Relations

Linear Recurrence Relations of Degree 2

- $a_{n+1} = f(n)a_n + g(n)a_{n-1}$ with non-constant coefficients $f(n)$ and $g(n)$.

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- Explore conditions on f and g such that the sequence generated obeys Benford's Law for all initial values.
- First solve the closed form of the sequence (a_n), then analyze its main term.

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- Define an auxiliary sequence $\{b_n\}_{n=1}^{\infty}$ by $b_n = a_{n+1} - \lambda(n)a_n$ for $n \geq 1$.
  ((a_n) recurrent of degree 2, so (b_n) of degree 1).

  Set $a_{n+1} - \lambda(n)a_n = \mu(n)(a_n - \lambda(n-1)a_{n-1})$ for $n \geq 2$.

- $a_{n+1} = (\lambda(n) + \mu(n))a_n - \mu(n)\lambda(n-1)a_{n-1}$, and compare the coefficients:

$$\begin{aligned} f(n) &= \lambda(n) + \mu(n) \\ g(n) &= -\lambda(n-1)\mu(n). \end{aligned}$$

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- We show that for any given pair of f and g, such λ and μ always exist.

Linear Recurrence Relations of Degree 2

- Recurrence relations of degree 1:

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$$\begin{aligned} a_{n+1} &= \lambda(n) a_n + b_n \\ b_n &= \mu(n) b_{n-1}. \end{aligned}$$

- $a_{n+1} = r(n)\left(1 + \sum_{k=3}^{n} \prod_{i=k}^{n} \frac{\lambda(i)}{\mu(i)} + \frac{a_2}{b_1} \prod_{i=2}^{n} \frac{\lambda(i)}{\mu(i)}\right)$, where $r(n) := b_1 \prod_{i=2}^{n} \mu(i)$.

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- Let $\frac{\lambda(n)}{\mu(n)}$ be a non-increasing function and $\lim_{n\to\infty} \frac{\lambda(n)}{\mu(n)} = 0$, then $\lim_{n\to\infty}(a_{n+1} - r(n)) = 0$.

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- $\lim_{n\to\infty} \frac{\lambda(n)}{\mu(n)} = 0$ implies $\lim_{n\to\infty} \frac{g(n)}{f(n)^2} = 0$.

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- Since (strong) Benford-ness is preserved under translation and dilation we let $r(n) = \prod_{i=1}^{n} \mu(i)$ for simplicity.

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- If $\mu(k) = k$, then $r(n) = n!$.
- If $\mu(k) = k^\alpha$ where $\alpha \in \mathbb{R}$, then $r(n) = (n!)^\alpha$.
- If $\mu(k) = \exp(\alpha h(k))$ where α is irrational and $h(k)$ is a monic polynomial, then $\log r(n) = \alpha \sum_{k=1}^{n} h(k)$.

Lemma

The sequence $\{\alpha p(n)\}$ is equidistributed mod 1 if $\alpha \notin \mathbb{Q}$ and $p(n)$ a monic polynomial.

Examples when f and g are random variables

- Take $\mu(n) \sim h(n)U_n$ where the U_n's are independent uniform distributions on $[0, 1]$, and $h(n)$ is a deterministic function in n such that $\prod_{i=1}^{n} h(i)$ is Benford.

  Then $r(n) = \prod_{i=1}^{n} h(i) \prod_{i=1}^{n} U_i$ is Benford.

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- Take $\mu(n) \sim \exp(U_n)$ where the U_n's are i.i.d. random variables. Then take logarithm and sum up $\log(\mu(n))$. Apply Central Limit Theorem and get a Gaussian distribution with increasing variance.

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Linear Recurrences of Higher Degree

- Use recurrence relation of degree 3 as an example. Similar main idea: reduce the degree.
- Define the sequence $\{a_n\}_{n=1}^{\infty}$ by $a_{n+1} \ = \ f_1(n)a_n + f_2(n)a_{n-1} + f_3(n)a_{n-2}$.
- Define an auxiliary sequence $(b_n)_{n=1}^{\infty}$ by $b_n \ = \ a_{n+1} - \lambda(n)a_n$. Then (b_n) is degree 2.

Appendix
Multiplicative Recurrence Relations

Generalization to Multiplicative Recurrence Relations

- Define sequence $(A_n)_{n=1}^{\infty}$ by the recurrence relation $A_{n+1} = A_n^{f(n)} A_{n-1}^{g(n)}$ with initial values A_1, A_2.

Generalization to Multiplicative Recurrence Relations

- Define sequence $(A_n)_{n=1}^{\infty}$ by the recurrence relation $A_{n+1} = A_n^{f(n)} A_{n-1}^{g(n)}$ with initial values A_1, A_2.

- Then the closed form of A_n is $A_n = A_2^{x_n} A_1^{y_n}$, where the exponents (x_n) and (y_n) satisfy the linear recurrence relations

$$\begin{aligned} x_{n+1} &= f(n)x_n + g(n)x_{n-1} \\ y_{n+1} &= f(n)y_n + g(n)y_{n-1} \end{aligned}$$

with initial values

$$\begin{aligned} x_1 &= 0, x_2 = 1, \\ y_1 &= 1, y_2 = 0. \end{aligned}$$

Generalization to Multiplicative Recurrence Relations

- Again, solve for (x_n) and (y_n) with auxiliary functions $\lambda(n)$ and $\mu(n)$.

Generalization to Multiplicative Recurrence Relations

- Again, solve for (x_n) and (y_n) with auxiliary functions $\lambda(n)$ and $\mu(n)$.

- Let $\lambda(n)$ and $\mu(n)$ satisfy $\lim_{n\to\infty} \frac{\lambda(n)}{\mu(n)} = 0$.

Generalization to Multiplicative Recurrence Relations

- Take the main terms:

$$x_{n+1} \rightarrow (x_2 - \lambda(1)x_1) \prod_{i=2}^{n} \mu(i) = \prod_{i=2}^{n} \mu(i),$$

$$y_{n+1} \rightarrow (y_2 - \lambda(1)y_1) \prod_{i=2}^{n} \mu(i) = -\lambda(1) \prod_{i=2}^{n} \mu(i).$$

Generalization to Multiplicative Recurrence Relations

- Take the main terms:

$$x_{n+1} \;\to\; (x_2 - \lambda(1)x_1)\prod_{i=2}^{n}\mu(i) \;=\; \prod_{i=2}^{n}\mu(i),$$
$$y_{n+1} \;\to\; (y_2 - \lambda(1)y_1)\prod_{i=2}^{n}\mu(i) \;=\; -\lambda(1)\prod_{i=2}^{n}\mu(i).$$

- By the closed form $A_n = A_2^{x_n}A_1^{y_n}$,

$$\begin{aligned}\log(A_{n+1}) &= x_n\log(A_2) + y_n\log(A_1)\\ &\to \Big(\prod_{i=2}^{n}\mu(i)\Big)(\log(A_2) - \lambda(1)\log(A_1))\end{aligned}$$

as $n \to \infty$.

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- (A_n) is a Benford sequence if the main term of $\log(A_n)$ is equidistributed mod 1.
- We choose μ and the initial values such that $(\log(A_2) - \lambda(1)\log(A_1))\prod_{i=2}^{n}\mu(i)$ is equidistributed mod 1.

Examples

Remeber: The sequence $\{\alpha p(n)\}$ is equidistributed mod 1 if $\alpha \notin \mathbb{Q}$ and $p(n)$ a monic polynomial.

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Our construction:

- $\frac{\log(A_2) - \lambda(1)\log(A_1)}{\mu(1)} =: \alpha \notin \mathbb{Q}$,
- $\mu(i) = \frac{p(i)}{p(i-1)}$ where $p(n)$ is a non-vanishing monic polynomial.

Open Questions and References

Open Questions

1. We mainly consider the case when $\frac{\lambda(n)}{\mu(n)} \to 0$ as $n \to \infty$. What about when $\frac{\lambda(n)}{\mu(n)} \to \infty$? In this case, there is no simple dominating term.

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2. Applications of recurrence relations when $f(n)$ and $g(n)$ are random variables?

Acknowledgement

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