Linear Recurrence Relations with Non-constant Coefficients and Benford’s Law

Mengxi Wang, Lily Shao
University of Michigan, Williams College

mengxiw@umich.edu
ls12@williams.edu

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Motivating Question: For a nice data set, such as the Fibonacci numbers, stock prices, street addresses of college employees and students, ..., what percent of the leading digits are 1?
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Answer: Benford’s law!
Examples with First Digit Bias

Fibonacci numbers
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Fibonacci numbers

Most common iPhone passcodes
Examples with First Digit Bias

- Fibonacci numbers
- Twitter users by # followers
- Most common iPhone passcodes
Examples with First Digit Bias

Fibonacci numbers

Twitter users by # followers

Most common iPhone passcodes

Distance of stars from Earth
Applications

- Analyzing round-off errors.
- Determining the optimal way to store numbers.
- Detecting tax and image fraud, and data integrity.
An Interesting Question

From previous works: sequences generated by linear recurrence relations with constant coefficients obey Benford’s Law.
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- Example: The Fibonacci Sequence 1, 2, 3, 5, 8, 13, 21, 34, . . . .
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- Example: The Fibonacci Sequence 1, 2, 3, 5, 8, 13, 21, 34, . . . .

- **Question**: Non-constant coefficients?
Outline

- Benford’s Law.
- Linear Recurrence relations (degree 2 and higher degree).
- Multiplicative Recurrence relations.
- Open problems and references.
Equidistribution and Benford’s Law
Benford’s Law

A set of numbers is **Benford** (base $b$) if the probability of observing a first digit of $d$ is $\log_b \left(1 + \frac{1}{d}\right)$. 

Strong Benford

A set of numbers is **Strong Benford** (base $b$) if the probability of observing a significand in $[1, s)$ is $\log_b(s)$. Then the probability of observing a significand in $[s, s+1)$ is $\log_b \left(1 + \frac{1}{s}\right)$. 

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Equidistribution and Benford’s Law

**Equidistribution**

\[ \{ y_n \}_{n=1}^{\infty} \] is equidistributed modulo 1 if probability \( y_n \mod 1 \in [a, b] \) tends to \( b - a \):

\[
\frac{\# \{ n \leq N : y_n \mod 1 \in [a, b] \}}{N} \to b - a.
\]

**Theorem**

\( \beta \notin \mathbb{Q}, \ n\beta \) is equidistributed mod 1.
Logarithms and Benford’s Law

Fundamental Equivalence

Data set \( \{x_i\} \) is Benford base \( B \) if \( \{y_i\} \) is equidistributed \( \text{mod} \ 1 \), where \( y_i = \log_B x_i \).
Logarithms and Benford’s Law

**Fundamental Equivalence**

Data set \( \{x_i\} \) is Benford base \( B \) if \( \{y_i\} \) is equidistributed mod 1, where \( y_i = \log_B x_i \).

\[
x = S_{10}(x) \cdot 10^k \text{ then } \log_{10} x = \log_{10} S_{10}(x) + k = \log_{10} S_{10}x \mod 1.
\]
Logarithms and Benford’s Law

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Logarithms and Benford’s Law

\[
\text{Prob(leading digit } d) = \log_{10}(d + 1) - \log_{10}(d) = \log_{10}\left(\frac{d+1}{d}\right) = \log_{10}\left(1 + \frac{1}{d}\right).
\]

Have Benford’s law \(\leftrightarrow\) mantissa (fractional part) of logarithms of data are uniformly distributed.
Examples

- Remember: $\beta \notin \mathbb{Q}$, $n\beta$ is equidistributed mod 1.
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- $2^n$ is Benford base 10 as $\log_{10} 2 \not\in \mathbb{Q}$.

- Fibonacci numbers are Benford base 10.

  Binet: $a_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n$. 
It suffices to analyze the main term of the sequence:

**Theorem**

If a sequence \( \{ a_n \} \) is Benford and \( \lim_{n \to \infty} b_n = a_n \) then \( \{ b_n \} \) is Benford as well.
Linear Recurrence Relations
Linear Recurrence Relations of Degree 2

- \( a_{n+1} = f(n)a_n + g(n)a_{n-1} \) with non-constant coefficients \( f(n) \) and \( g(n) \).
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- Explore conditions on \( f \) and \( g \) such that the sequence generated obeys Benford’s Law for all initial values.
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- Explore conditions on \( f \) and \( g \) such that the sequence generated obeys Benford’s Law for all initial values.

- First solve the closed form of the sequence \( (a_n) \), then analyze its main term.
Linear Recurrence Relations of Degree 2

To solve for the closed form of the sequence:

- Main idea: reduce the degree of recurrence.
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- **Main idea:** reduce the degree of recurrence.

- Define an auxiliary sequence \( \{b_n\}_{n=1}^{\infty} \) by
  \[
  b_n = a_{n+1} - \lambda(n)a_n \quad \text{for} \quad n \geq 1.
  \]
  ((\( a_n \)) recurrent of degree 2, so (\( b_n \)) of degree 1).

Set \( a_{n+1} - \lambda(n)a_n = \mu(n)(a_n - \lambda(n-1)a_{n-1}) \) for \( n \geq 2 \).
\[ a_{n+1} = (\lambda(n) + \mu(n))a_n - \mu(n)\lambda(n-1)a_{n-1}, \]
and compare the coefficients:

\[
\begin{align*}
    f(n) &= \lambda(n) + \mu(n) \\
    g(n) &= -\lambda(n-1)\mu(n).
\end{align*}
\]
\[ a_{n+1} = (\lambda(n) + \mu(n))a_n - \mu(n)\lambda(n-1)a_{n-1}, \] and compare the coefficients:

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\end{align*}
\]

We show that for any given pair of \( f \) and \( g \), such \( \lambda \) and \( \mu \) always exist.
Linear Recurrence Relations of Degree 2

- Recurrence relations of degree 1:

\[ a_{n+1} = \lambda(n)a_n + b_n \]
\[ b_n = \mu(n)b_{n-1}. \]
Linear Recurrence Relations of Degree 2

- Recurrence relations of degree 1:
  \[ a_{n+1} = \lambda(n)a_n + b_n \]
  \[ b_n = \mu(n)b_{n-1}. \]

- \[ a_{n+1} = r(n) \left( 1 + \sum_{k=3}^{n} \prod_{i=k}^{n} \frac{\lambda(i)}{\mu(i)} + \frac{a_2}{b_1} \prod_{i=2}^{n} \frac{\lambda(i)}{\mu(i)} \right), \] where \[ r(n) := b_1 \prod_{i=2}^{n} \mu(i). \]
Asymptotic analysis: show the main term dominates for suitable choices of $\mu$ and $\lambda$. 
Linear Recurrence Relations of Degree 2

- Asymptotic analysis: show the main term dominates for suitable choices of $\mu$ and $\lambda$.

- Let $\frac{\lambda(n)}{\mu(n)}$ be a non-increasing function and
  $$\lim_{n \to \infty} \frac{\lambda(n)}{\mu(n)} = 0,$$
  then
  $$\lim_{n \to \infty} (a_{n+1} - r(n)) = 0.$$
Asymptotic analysis: show the main term dominates for suitable choices of $\mu$ and $\lambda$.

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$$\lim_{n \to \infty} \frac{\lambda(n)}{\mu(n)} = 0$$
implies
$$\lim_{n \to \infty} \frac{g(n)}{f(n)^2} = 0.$$
Benford-ness of the Main Term

- Main term of $a_{n+1}$ is $r(n) = b_1 \prod_{i=2}^{n} \mu(i)$. 
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- Since (strong) Benford-ness is preserved under translation and dilation we let $r(n) = \prod_{i=1}^{n} \mu(i)$ for simplicity.
Examples when $f$ and $g$ are functions

- If $\mu(k) = k$, then $r(n) = n!$. 
Examples when \( f \) and \( g \) are functions

- If \( \mu(k) = k \), then \( r(n) = n! \).

- If \( \mu(k) = k^\alpha \) where \( \alpha \in \mathbb{R} \), then \( r(n) = (n!)^\alpha \).
Examples when $f$ and $g$ are functions

- If $\mu(k) = k$, then $r(n) = n!$.

- If $\mu(k) = k^\alpha$ where $\alpha \in \mathbb{R}$, then $r(n) = (n!)^\alpha$.

- If $\mu(k) = \exp(\alpha h(k))$ where $\alpha$ is irrational and $h(k)$ is a monic polynomial, then $\log r(n) = \alpha \sum_{k=1}^{n} h(k)$.

**Lemma**

The sequence $\{\alpha p(n)\}$ is equidistributed mod 1 if $\alpha \notin \mathbb{Q}$ and $p(n)$ a monic polynomial.
Examples when $f$ and $g$ are random variables

- Take $\mu(n) \sim h(n) U_n$ where the $U_n$’s are independent uniform distributions on $[0, 1]$, and $h(n)$ is a deterministic function in $n$ such that $\prod_{i=1}^{n} h(i)$ is Benford.

Then $r(n) = \prod_{i=1}^{n} h(i) \prod_{i=1}^{n} U_i$ is Benford.
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- Take $\mu(n) \sim \exp(U_n)$ where the $U_n$’s are i.i.d. random variables. Then take logarithm and sum up $\log(\mu(n))$. Apply Central Limit Theorem and get a Gaussian distribution with increasing variance.
Use recurrence relation of degree 3 as an example. Similar main idea: reduce the degree.
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Define the sequence \( \{a_n\}_{n=1}^\infty \) by
\[
a_{n+1} = f_1(n)a_n + f_2(n)a_{n-1} + f_3(n)a_{n-2}.
\]
Linear Recurrences of Higher Degree

- Use recurrence relation of degree 3 as an example. Similar main idea: reduce the degree.

- Define the sequence \( \{a_n\}_{n=1}^{\infty} \) by
  \[
an_{n+1} = f_1(n)a_n + f_2(n)a_{n-1} + f_3(n)a_{n-2}.
  \]

- Define an auxiliary sequence \( (b_n)_{n=1}^{\infty} \) by
  \[
b_n = a_{n+1} - \lambda(n)a_n. \text{ Then } (b_n) \text{ is degree 2.}
  \]
Appendix
Multiplicative Recurrence Relations
Generalization to Multiplicative Recurrence Relations

Define sequence \((A_n)_{n=1}^{\infty}\) by the recurrence relation

\[ A_{n+1} = A_n^{f(n)} A_{n-1}^{g(n)} \]

with initial values \(A_1, A_2\).
Generalization to Multiplicative Recurrence Relations

Define sequence \((A_n)_{n=1}^\infty\) by the recurrence relation
\[ A_{n+1} = A_n^{f(n)} A_{n-1}^{g(n)} \]
with initial values \(A_1, A_2\).

Then the closed form of \(A_n\) is \(A_n = A_2^{x_n} A_1^{y_n}\), where the exponents \((x_n)\) and \((y_n)\) satisfy the linear recurrence relations
\[
\begin{align*}
x_{n+1} &= f(n)x_n + g(n)x_{n-1} \\
y_{n+1} &= f(n)y_n + g(n)y_{n-1}
\end{align*}
\]
with initial values
\[
\begin{align*}
x_1 &= 0, \ x_2 = 1, \\
y_1 &= 1, \ y_2 = 0.
\end{align*}
\]
Generalization to Multiplicative Recurrence Relations

- Again, solve for \((x_n)\) and \((y_n)\) with auxiliary functions \(\lambda(n)\) and \(\mu(n)\).
Generalization to Multiplicative Recurrence Relations

- Again, solve for \((x_n)\) and \((y_n)\) with auxiliary functions \(\lambda(n)\) and \(\mu(n)\).

- Let \(\lambda(n)\) and \(\mu(n)\) satisfy \(\lim_{n \to \infty} \frac{\lambda(n)}{\mu(n)} = 0\).
Generalization to Multiplicative Recurrence Relations

- Take the main terms:

\[ x_{n+1} \to (x_2 - \lambda(1)x_1) \prod_{i=2}^{n} \mu(i) = \prod_{i=2}^{n} \mu(i), \]

\[ y_{n+1} \to (y_2 - \lambda(1)y_1) \prod_{i=2}^{n} \mu(i) = -\lambda(1) \prod_{i=2}^{n} \mu(i). \]
Generalization to Multiplicative Recurrence Relations

- Take the main terms:
  \[ x_{n+1} \rightarrow (x_2 - \lambda(1)x_1) \prod_{i=2}^{n} \mu(i) = \prod_{i=2}^{n} \mu(i), \]
  \[ y_{n+1} \rightarrow (y_2 - \lambda(1)y_1) \prod_{i=2}^{n} \mu(i) = -\lambda(1) \prod_{i=2}^{n} \mu(i). \]

- By the closed form \( A_n = A_2^{x_n} A_1^{y_n}, \)
  \[ \log(A_{n+1}) = x_n \log(A_2) + y_n \log(A_1) \]
  \[ \rightarrow (\prod_{i=2}^{n} \mu(i))(\log(A_2) - \lambda(1) \log(A_1)) \]

as \( n \rightarrow \infty. \)
Benford-ness of the main term

- \((A_n)\) is a Benford sequence if the main term of \(\log(A_n)\) is equidistributed mod 1.
(\(A_n\)) is a Benford sequence if the main term of \(\log(A_n)\) is equidistributed mod 1.

We choose \(\mu\) and the initial values such that

\[
(\log(A_2) - \lambda(1) \log(A_1)) \prod_{i=2}^{n} \mu(i)
\]

is equidistributed mod 1.
Examples

Remember: The sequence $\{\alpha p(n)\}$ is equidistributed mod 1 if $\alpha \notin \mathbb{Q}$ and $p(n)$ a monic polynomial.
Examples

Remeber: The sequence \( \{\alpha p(n)\} \) is equidistributed mod 1 if \( \alpha \notin \mathbb{Q} \) and \( p(n) \) a monic polynomial.

Our construction:

1. \[ \frac{\log(A_2) - \lambda(1) \log(A_1)}{\mu(1)} =: \alpha \notin \mathbb{Q}, \]

2. \[ \mu(i) = \frac{p(i)}{p(i-1)} \] where \( p(n) \) is a non-vanishing monic polynomial.
Open Questions and References
We mainly consider the case when \( \frac{\lambda(n)}{\mu(n)} \to 0 \) as \( n \to \infty \). What about when \( \frac{\lambda(n)}{\mu(n)} \to \infty \)? In this case, there is no simple dominating term.
Open Questions

1. We mainly consider the case when \( \frac{\lambda(n)}{\mu(n)} \rightarrow 0 \) as \( n \rightarrow \infty \). What about when \( \frac{\lambda(n)}{\mu(n)} \rightarrow \infty \)? In this case, there is no simple dominating term.

2. Applications of recurrence relations when \( f(n) \) and \( g(n) \) are random variables?
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References


