Eigenvalue Distributions of Kronecker Random Matrices

Noah Luntzlara (nluntzla@umich.edu) & Mengxi Wang (mengxiw@umich.edu); Advisor: Steven J. Miller

Number Theory and Probability Group - SMALL 2018 - Williams College

1. Background and Introduction

Random matrix theory investigates the distribution of eigenvalues of random matrix ensembles. It has successfully been used as a model for applications in number theory and nuclear physics, among others.

Eigenvalue-Trace Lemma: Let $\mathbf{A}$ be an $N \times N$ matrix with eigenvalues $\lambda_i(\mathbf{A})$. Then

$$\text{Trace}(\mathbf{A}^k) \;=\; \sum_{i=1}^{N} \lambda_i(\mathbf{A})^k,$$

where

$$\text{Trace}(\mathbf{A}^k) \;=\; \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}.$$

Normalization: We adjust the scale of all our eigenvalue distributions so that they have variance $\sigma^2 = 1$. By the Eigenvalue-Trace Lemma, $\sum_{i=1}^{N} \lambda_i(\mathbf{A})^2 = \text{Trace}(\mathbf{A}^2) \sim N^2$, so to re-scale the eigenvalues we divide them by $\sqrt{N}$.

Eigenvalue Distribution: To each matrix $\mathbf{A}$, assign a probability measure

$$\mu_{\mathbf{A},N}(x) \;=\; \frac{1}{N} \sum_{i=1}^{N} \delta\Big(x - \frac{\lambda_i(\mathbf{A})}{\sqrt{N}}\Big).$$

Then

$$\int_a^b \mu_{\mathbf{A},N}(x)dx \;=\; \frac{\#\{\lambda_i : \frac{\lambda_i(\mathbf{A})}{\sqrt{N}} \in [a,b]\}}{N},$$

$$k^{\text{th}} \text{ moment} \;=\; \int_{-\infty}^{\infty} x^k \mu_{\mathbf{A},N}(x)dx \;=\; \frac{\sum_{i=1}^{N} \lambda_i(\mathbf{A})^k}{N^{\frac{k}{2}+1}}$$

$$= \frac{\text{Trace}(\mathbf{A}^k)}{N^{\frac{k}{2}+1}} \;=\; \frac{\sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}}{N^{\frac{k}{2}+1}}.$$

Wigner's Semicircle Law: In the ensemble of $N \times N$ real Wigner matrices, for almost all matrices $\mathbf{A}$ as N approaches infinity

$$\mu_{\mathbf{A},N} \to \begin{cases} \frac{1}{2\pi}\sqrt{4 - x^2} & \text{if } |x| \le 2 \\ 0 & \text{if } |x| > 2 \end{cases}$$

where $\mu_{\mathbf{A},N}$ is the probability measure

$$\mu_{\mathbf{A},N} = \frac{1}{N} \sum_{i=1}^{N} \delta\left(x - \frac{\lambda_i}{\sqrt{N}}\right)$$

for $\{\lambda_i\}_{i=1}^{N}$ the eigenvalues of $\mathbf{A}$.

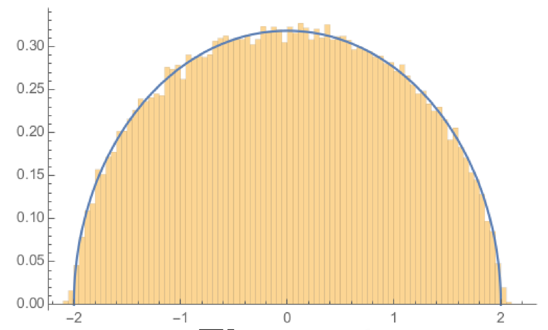

Figure 1.

Normalized eigenvalues from 500 100×100 real symmetric matrices.

2. Kronecker Products of Random Matrices

Definition: The Kronecker product of an $n \times n$ matrix $\mathbf{A}$ and an $m \times m$ matrix $\mathbf{B}$ is the $nm \times nm$ block matrix

$$\mathbf{A} \otimes \mathbf{B} \;=\; \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}\mathbf{B} & a_{n2}\mathbf{B} & \cdots & a_{nn}\mathbf{B} \end{bmatrix}.$$

The Kronecker product has the property that if

$\mathbf{A}$ has eigenvalues λ_i, $1 \le i \le n$ and
$\mathbf{B}$ has eigenvalues μ_j, $1 \le j \le m$, then
$\mathbf{A} \otimes \mathbf{B}$ has eigenvalues $\lambda_i \mu_j$, $1 \le i \le n$, $1 \le j \le m$.

This property implies the following theorem.

Theorem: Let $\mathbf{A}$ be chosen at random from some matrix ensemble and $\mathbf{B}$ be chosen at random from a possibly different ensemble. If the average moments of the eigenvalue distributions of $\mathbf{A}$ and $\mathbf{B}$ all exist, then the average k^{th} moment of $\mathbf{A} \otimes \mathbf{B}$ is the product of the average k^{th} moments of $\mathbf{A}$ and $\mathbf{B}$.

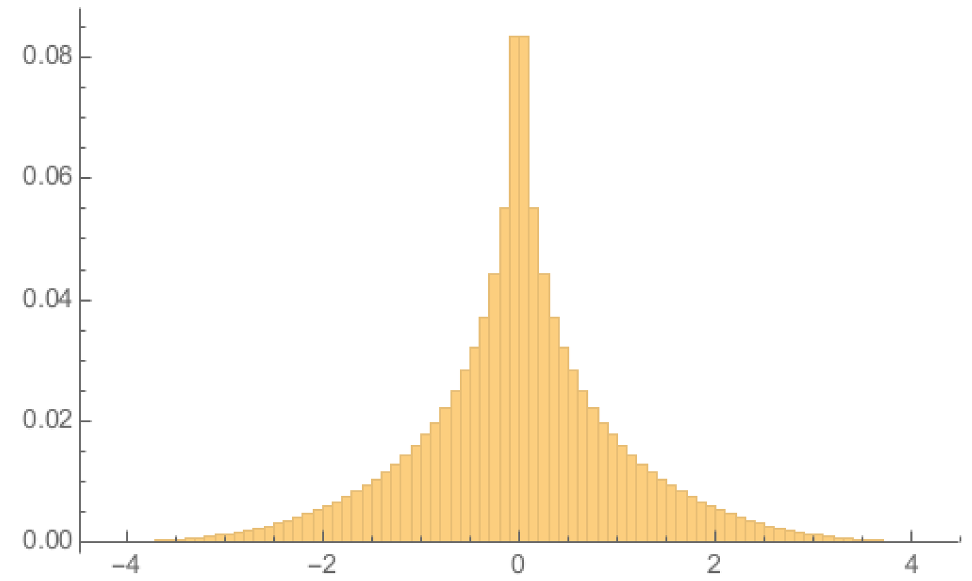

Figure 2. Normalized eigenvalue distribution of the Kronecker product of two independent real symmetric matrices.

3. "Disco" Matrices

Definition: The disco matrix of two independent $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ is the $2n \times 2n$ block matrix

$$\text{Disco}(\mathbf{A},\mathbf{B}) \;=\; \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix} \;=:\; \mathbf{D}.$$

We normalize the eigenvalues of Disco$(\mathbf{A},\mathbf{B})$, dividing by $2\sqrt{N}$. We explore the eigenvalue distributions of Disco matrices through the method of moments. We would like to compute the trace of powers of $\mathbf{D}$ in order to apply the Eigenvalue-Trace Lemma. Diagonalizing $\mathbf{D}$ gives

$$\mathbf{D}^k \;=\; \begin{bmatrix} \mathbf{I}/2 & \mathbf{I}/2 \\ \mathbf{I}/2 & -\mathbf{I}/2 \end{bmatrix} \begin{bmatrix} (\mathbf{A}+\mathbf{B})^k & 0 \\ 0 & (\mathbf{A}-\mathbf{B})^k \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}.$$

So

$$\text{Trace}(\mathbf{D}^k) \;=\; 2 \sum_{\substack{l=0 \\ l:\text{even}}}^{k} \sum_{i_1+\cdots+i_p=k-l} \sum_{j_1+\cdots+j_p=l} \text{Trace}(\mathbf{A}^{i_1}\mathbf{B}^{j_1} \cdots \mathbf{A}^{i_p}\mathbf{B}^{j_p}).$$

Note: The normalized eigenvalue distribution of Disco$(\mathbf{A},\mathbf{B})$ and that of Disco$(\mathbf{B},\mathbf{A})$ are the same.

Theorem: Suppose $\mathbf{A}$ and $\mathbf{B}$ are independent $n \times n$ matrices chosen from the same ensemble. The the normalized eigenvalue distribution of Disco$(\mathbf{A},\mathbf{B})$ is the same as that of $\mathbf{A}$ or $\mathbf{B}$.

Conjecture: Suppose $\mathbf{A}$ and $\mathbf{B}$ are independent $n \times n$ matrices chosen from different ensembles. Then if the average k^{th} moments of $\mathbf{A}$ and $\mathbf{B}$ exist, the average k^{th} moment of Disco$(\mathbf{A},\mathbf{B})$ lies between them.

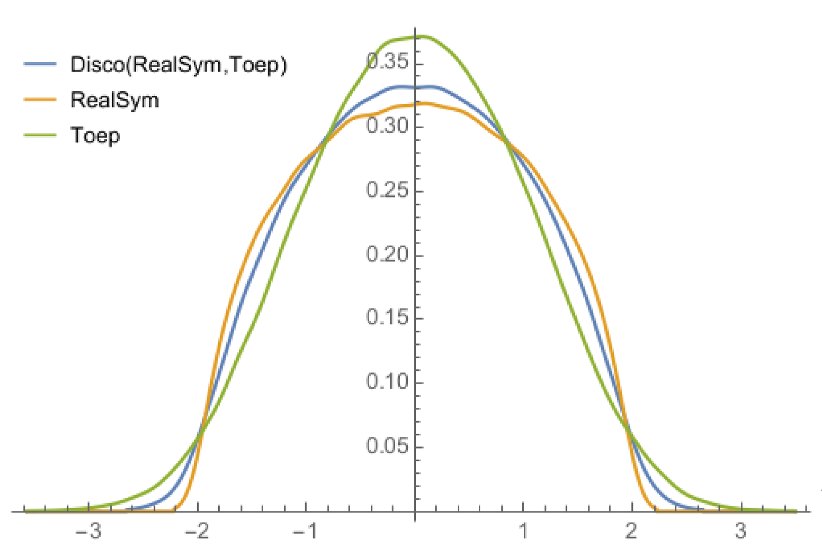

Figure 3. Real Symmetric, Toeplitz, and their Disco

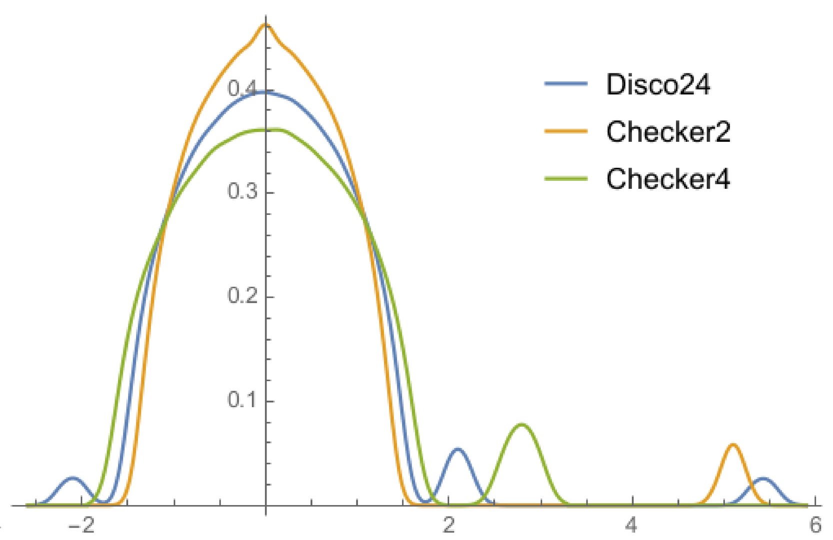

Figure 4. Checkerboard with parameters 2 and 4, and their Disco

Remark: Although these statements tell us that the limiting distributions of eigenvalues should be well behaved under the operation $D(\cdot,\cdot)$, lower order terms have been observed, such as the "blip" in the distribution arising from the checkerboard ensemble. Numerical evidence shows that these effects are not as predictable under Disco$(\cdot,\cdot)$ (see Figure 4).

4. Acknowledgements

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