1. Background and Introduction

Random matrix theory investigates the distribution of eigenvalues of random matrix ensembles. It has successfully been used as a model for applications in number theory and nuclear physics, among others.

**Eigenvalue-Trace Lemma**: Let \( A \) be an \( N \times N \) matrix with eigenvalues \( \lambda_i(A) \). Then

\[
\text{Trace}(A^k) = \sum_{i=1}^{N} \lambda_i(A)^k,
\]

where

\[
\text{Trace}(A^k) = \sum_{i,j=1}^{N} a_{ij} a_{ji} \cdots a_{ik}.
\]

**Normalization**: We adjust the scale of all our eigenvalue distributions so that they have variance \( \sigma^2 = 1 \). By the Eigenvalue-Trace Lemma, \( \sum_{i=1}^{N} \lambda_i(A)^2 = \text{Trace}(A^2) \sim N^2 \), so to re-scale the eigenvalues we divide them by \( \sqrt{N} \).

**Eigenvalue Distribution**: To each matrix \( A \), assign a probability measure

\[
\mu_{A,N}(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - \frac{\lambda_i(A)}{\sqrt{N}}),
\]

Then

\[
\int_{a}^{b} \mu_{A,N}(x)dx = \frac{\# \{ \lambda_i : \frac{\lambda_i}{\sqrt{N}} \in [a,b] \}}{N},
\]

\( k \)-th moment

\[
= \int_{-\infty}^{\infty} x^k \mu_{A,N}(x)dx = \frac{\sum_{i=1}^{N} x^k \lambda_i(A)^k}{N^{k+1}} = \frac{\text{Trace}(A^k)}{N^{k+1}} = \sum_{i=1}^{N} \cdots \sum_{i=1}^{N} a_{ij_1} a_{ji_2} \cdots a_{ji_k}.
\]

**Wigner’s Semicircle Law**: In the ensemble of \( N \times N \) real Wigner matrices, for almost all matrices \( A \) as \( N \) approaches infinity

\[
\mu_{A,N} \to 2 \sqrt{1 - x^2} \quad \text{if} \quad |x| \leq 2
\]

\[
0 \quad \text{if} \quad |x| > 2
\]

where \( \mu_{A,N} \) is the probability measure

\[
\mu_{A,N} = \frac{1}{500000} \sum_{i=1}^{500000} \delta(x - \frac{\lambda_i(A)}{\sqrt{N}})
\]

for \( \{ \lambda_i \}_{i=1}^{N} \) the eigenvalues of \( A \).

2. Kronecker Products of Random Matrices

**Definition**: The Kronecker product of an \( n \times n \) matrix \( A \) and an \( m \times m \) matrix \( B \) is the \( nm \times nm \) block matrix

\[
A \otimes B = \begin{bmatrix}
0_{1,n} \otimes B & \cdots & 0_{m,n} \otimes B \\
\vdots & \ddots & \vdots \\
0_{1,n} \otimes B & \cdots & 0_{m,n} \otimes B
\end{bmatrix}.
\]

The Kronecker product has the property that if \( A \) has eigenvalues \( \lambda_i \), \( 1 \leq i \leq n \) and \( B \) has eigenvalues \( \mu_j \), \( 1 \leq j \leq m \), then

\[
A \otimes B \text{ has eigenvalues } \lambda_i \mu_j, 1 \leq i \leq n, 1 \leq j \leq m.
\]

This property implies the following theorem.

**Theorem**: Let \( A \) be chosen at random from some matrix ensemble and \( B \) be chosen at random from a possibly different ensemble. If the average moments of the eigenvalue distributions of \( A \) and \( B \) all exist, then the average \( k \)-th moment of \( A \otimes B \) is the product of the average \( k \)-th moments of \( A \) and \( B \).

**Remark**: Although these statements tell us that the limiting distributions of eigenvalues should be well behaved under the operation \( D(\cdot,\cdot) \), lower order terms have been observed, such as the “blip” in the distribution arising from the checkerboard ensemble. Numerical evidence shows that these effects are not as predictable under \( D(\cdot,\cdot) \) (see Figure 5).

3. “Disco” Matrices

**Definition**: The disco matrix of two independent \( n \times n \) matrices \( A \) and \( B \) is the \( 2n \times 2n \) block matrix

\[
\text{Disco}(A, B) = \begin{bmatrix}
A & B \\
B & A
\end{bmatrix} = D.
\]

We normalize the eigenvalues of \( \text{Disco}(A, B) \), dividing by \( 2\sqrt{N} \). We explore the eigenvalue distributions of Disco matrices through the method of moments. We would like to compute the trace of powers of \( D \) in order to apply the Eigenvalue-Trace Lemma. Diagonalizing \( D \) gives

\[
D^k = \begin{bmatrix}
I/2 & I/2 \\
I/2 & I/2
\end{bmatrix} \begin{bmatrix}
(A+B)^k & 0 \\
0 & (A-B)^k
\end{bmatrix} \begin{bmatrix}
I/2 & -I/2 \\
I/2 & I/2
\end{bmatrix}.
\]

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