Phase Transitions in the Distribution of Missing Sums

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Ohio State University

Introduction

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Definition

A finite set of integers. A is called **sum-dominated** or MSTD (more-sum-than-difference) if $|A + A| > |A - A|$, **balanced** if $|A + A| = |A - A|$ and **difference-dominated** if $|A + A| < |A - A|$.

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- Each pair (x, y), $x \neq y$ gives two differences: $x - y \neq y - x$, but only one sum $x + y$.
- However, sets A with $|A+A| > |A-A|$ do exist!

Examples

- Conway: $A_1 = \{0, 2, 3, 4, 7, 11, 12, 14\}$
- Marica: $A_2 = \{0, 1, 2, 4, 7, 8, 12, 14, 15\}$
- Pigarev and Freiman: $A_3 = \{0, 1, 2, 4, 5, 9, 12, 13, 14, 16, 17, 21, 24, 25, 26, 28, 29\}$
- Hegarty: $A_4 = \{0, 1, 2, 4, 5, 9, 12, 13, 17, 20, 21, 22, 24, 25, 29, 32, 33, 37, 40, 41, 42, 44, 45\}$

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- Hegarty: $A_4 = \{0, 1, 2, 4, 5, 9, 12, 13, 17, 20, 21, 22, 24, 25, 29, 32, 33, 37, 40, 41, 42, 44, 45\}$
- Hegarty proved that the smallest cardinality of MSTD sets is 8.

Martin and Obryant '06

Theorem

Consider $I_n = \{0, 1, ..., n-1\}$. The proportion of MSTD subsets of I_n is bounded below by a positive constant $c \approx 2 \cdot 10^{-7}$.

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- Probabilistic method.

Results

Overview

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- The distribution has some strange behavior:
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  - Against missing certain number of sums.

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- Let $I_n = \{0, 1, 2, ..., n-1\}$.
- Form $S \subseteq I_n$ **randomly with probability p of picking an element in I_n**. ($q = 1 - p$: the probability of not chosing an element.)
- $B_n = (I_n + I_n) \backslash (S + S)$ is the set of missing sums, $|B_n|$: the number of missing sums.

Distribution of Missing Sums

- Fix $p \in (0,1)$, study $\mathbb{P}(|B| = k) = \lim_{n\to\infty} \mathbb{P}(|B_n| = k)$. (Zhao proved that the limit exists.)

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Divot

For some $k \geq 1$, if
$\mathbb{P}(|B| = k-1) > \mathbb{P}(|B| = k) < \mathbb{P}(|B| = k+1)$, then the distribution of sums has a **divot** at k.

Example of Divot

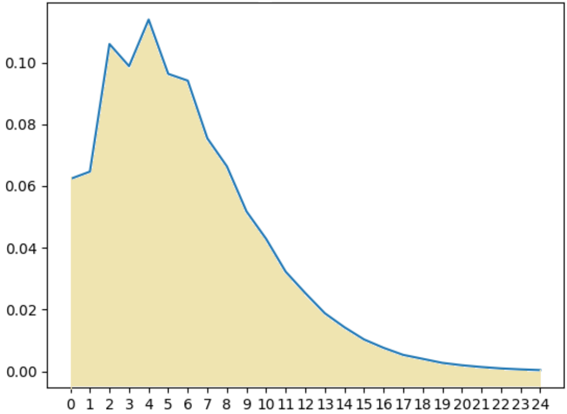

Figure: Frequency of the number of missing sums for subsets of $\{0, 1, 2, ..., 400\}$ by simulating 1,000,000 subsets with $p = 0.6$.

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- Interesting itself: two-bump distribution.

Numerical Analysis for $p = 1/2$

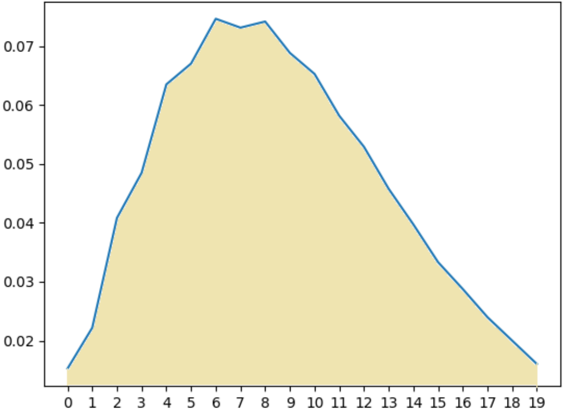

Figure: Frequency of the number of missing sums for all subsets of $\{0, 1, 2, ..., 25\}$.

Lazarev-Miller-O'Bryant '11

Divot at 7

For $p = 1/2$, there is a divot at 7, i.e.
$\mathbb{P}(|B| = 6) > \mathbb{P}(|B| = 7) < \mathbb{P}(|B| = 8)$.

Question

Existence of Divots

For a fixed different value of p, are there other divots?

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Answer: Yes!

Numerical analysis for different $p \in (0, 1)$: $p = 0.6$

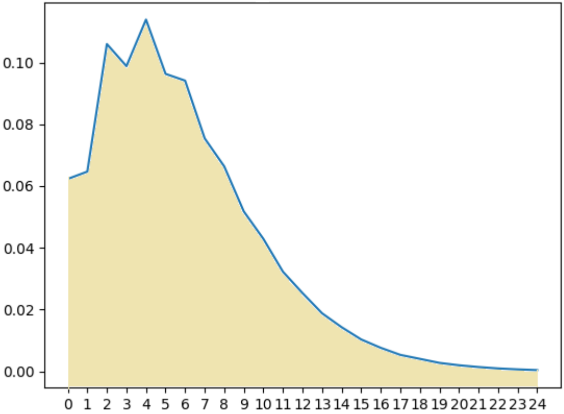

Figure: Distribution of $|B| = k$ by simulating 1,000,000 subsets of $\{0, 1, 2, \ldots, 400\}$ with $p = 0.6$.

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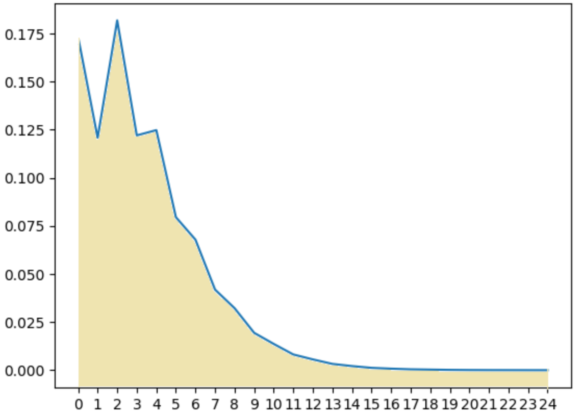

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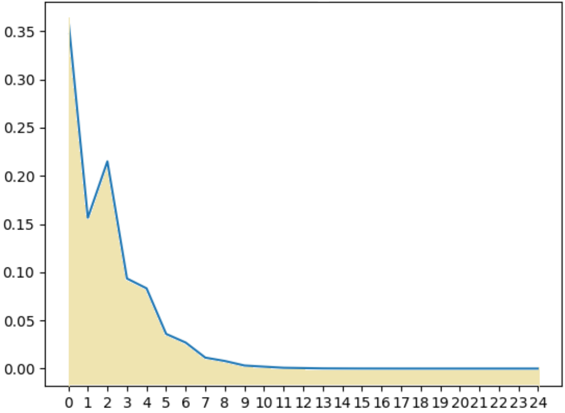

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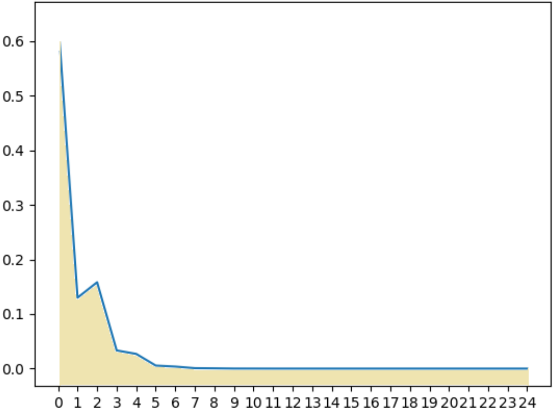

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Main Result

Chu-Luntzlara-Miller-Shao-Xu

For $p \geq 0.68$, there is a divot at 1, i.e.
$\mathbb{P}(|B| = 0) > \mathbb{P}(|B| = 1) < \mathbb{P}(|B| = 2)$.

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For $p \geq 0.68$, there is a divot at 1, i.e.
$\mathbb{P}(|B| = 0) > \mathbb{P}(|B| = 1) < \mathbb{P}(|B| = 2)$.

- Empirical evidence predicts the value of p such that the divot at 1 starts to exist is between 0.6 and 0.7.

Sketch of Proof

Key Ideas

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- Establish lower bounds T_0 and T_2 for $\mathbb{P}(|B| = 0)$ and $\mathbb{P}(|B| = 2)$, respectively.
- Find values of p such that $T_2 > T^1 < T_0$.

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- Fringe analysis is enough to find good lower bounds and upper bounds for $\mathbb{P}(|B| = k)$.

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- Similar notations applied for R.

Upper Bound

Given $0 \le k \le 30$,

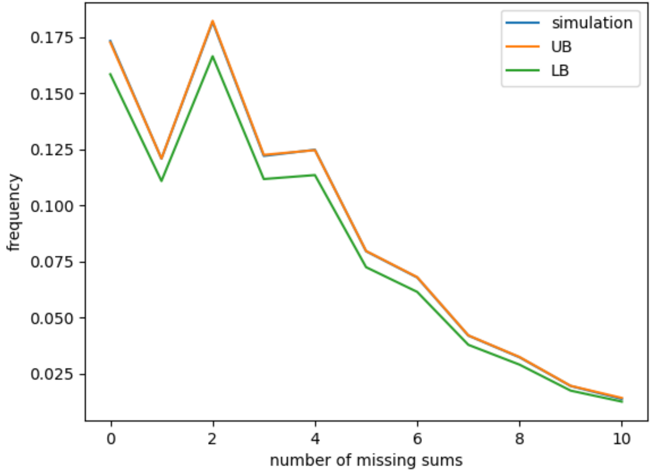
$$\mathbb{P}(|B| = k) \le \sum_{i=0}^{k} \mathbb{P}(L_i) P(L_{k-i}) + \frac{2(2q - q^2)^{15}(3q - q^2)}{(1 - q)^2}. \tag{1}$$

Lower Bound

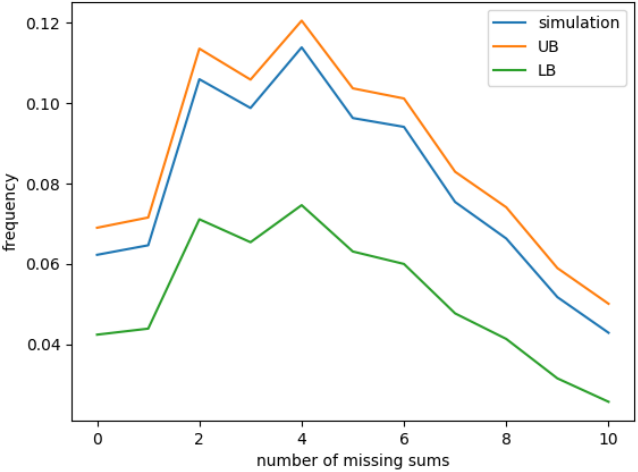
Given $0 \leq k \leq 30$,

$$\mathbb{P}(|B| = k) \geq \sum_{i=0}^{k} \Big[1 - (a-2)(q^{\tau(L_i^a)} + q^{\tau(L_{k-i}^a)}) - \frac{1+q}{(1-q)^2}(q^{\min L_i^a} + q^{\min L_{k-i}^a}) \Big] \mathbb{P}(L_i^a)\mathbb{P}(L_{k-i}^a). \quad (2)$$

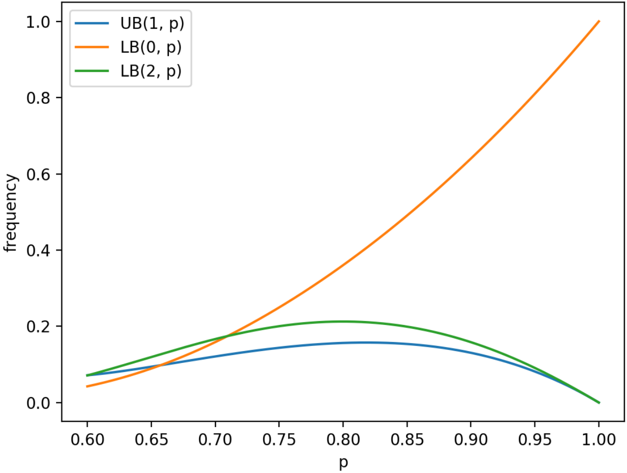
Our Bounds Are Sharp ($p \geq 0.7$)



Our Bounds Are Bad ($p \leq 0.6$)



Divot at 1


UB(1, p)
LB(0, p)
LB(2, p)
frequency
1.0
0.8
0.6
0.4
0.2
0.0
0.60
0.65
0.70
0.75
0.80
0.85
0.90
0.95
1.00
p

Future Research

Question

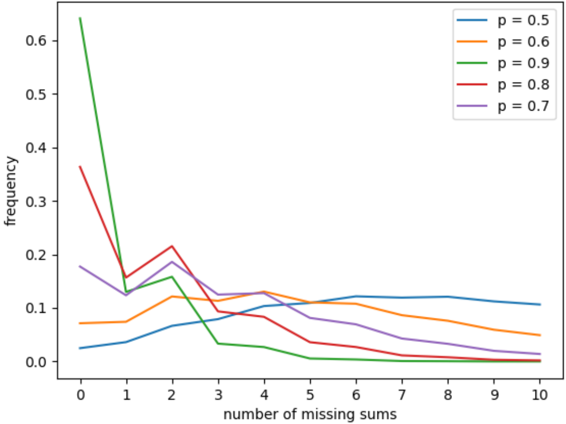


Figure: Shift of Divots

Question

Conjecture

There are no divots at even numbers.

Question

Is there a value of p such that there are no divots?

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THANK YOU!