Distributions of Primes in Number Fields and the L-function Ratios Conjecture

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- We say an ideal $\mathfrak{p}$ of a ring R is **prime** if it is not equal to R and $ab \in \mathfrak{p} \implies a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

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- To an ideal $\mathfrak{p} = \langle a + bi \rangle \subset \mathbb{Z}[i]$ we can associate an angle $\theta_{\mathfrak{p}}$ by
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- $\theta_{\mathfrak{p}}$ determined up to rotation by $\pi/2$.
- One can then study the smooth count of angles of prime ideals lying in a certain window:

$$\psi_{K,X}(\theta) = \sum_{\mathfrak{a} \subset \mathbb{Z}[i]} \Phi\left(\frac{N(\mathfrak{a})}{X}\right) \Lambda(\mathfrak{a}) F_K(\theta_{\mathfrak{a}} - \theta).$$

  - ◇ Φ a smooth compactly supported function
  - ◇ Λ a generalization of Von-Mangoldt
  - ◇ F_K detects angles of size $2\pi/K$.

History

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- SMALL 2017 REU: Calculated

$$\mathrm{Var}(\psi_{K,X}) = \frac{1}{2\pi} \int_0^{2\pi} \left| \psi_{K,X}(\theta) - \langle \psi_{K,X} \rangle \right|^2 d\theta.$$

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- Let $\mathcal{U}$ be the number of units in $\mathbb{Q}(\sqrt{-d})$
- To each ideal $\mathfrak{p} = \langle a + b\alpha_d \rangle$ we may associate the angle $\theta_{\mathfrak{p}}$ defined by

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- Thus, an analogous $\psi_{K,X}(\theta)$ can be defined and the same questions can be asked in this scenario: we are interested in studying $\mathrm{Var}(\psi_{K,X})$ in $\mathbb{Z}[\alpha_d]$.

L-functions

L-functions

An **L-function** is defined by a series of the form

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- $L(s,f)$ has an **Euler product**
- Usually convergent on a half-plane

L-functions

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- $\Lambda(s,f)$ satisfies a **functional equation** of the form

$$\Lambda(s,f) = \varepsilon_f \Lambda(1-s,f) \text{ with } \varepsilon_f = \pm 1$$

Example 1: Riemann zeta function

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$$\text{Functional equation: } \xi(s) = \xi(1-s)$$

Hecke characters and L-functions

A **Hecke character** χ is a homomorphism

$$\chi : I \to \mathbb{C}^*$$

where

- I is a multiplicative group of fractional ideals of a field
- $\mathbb{C}^*$ is the multiplicative group of complex numbers
- χ satisfies certain properties

Hecke characters and L-functions

If $\chi : I \to \mathbb{C}^*$ is a Hecke character, then a Hecke L-function is defined by a series of the form

$$L(s,\chi) = \sum_{\mathfrak{a} \in I} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}$$

where $N(\mathfrak{a})$ is the ideal norm of $\mathfrak{a}$

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Define Hecke character $\chi^{\mathcal{U}k}$ on ideals $\mathfrak{a} = \langle a + b\alpha_d \rangle$ of $\mathbb{Z}[\alpha_d]$ by

$$\chi^{\mathcal{U}k}(\mathfrak{a}) = \left(\frac{a + b\alpha_d}{|a + b\alpha_d|}\right)^{\mathcal{U}k} = e^{i\mathcal{U}k\theta_{\mathfrak{a}}}$$

Hecke L-function

Our main L-function of interest:

$$L_k(s) = \sum_{\mathfrak{a} \subset \mathbb{Z}[\alpha_d]} \frac{\chi^{\mathcal{U}k}(\mathfrak{a})}{N(\mathfrak{a})^s} = \sum_{\mathfrak{a} \subset \mathbb{Z}[\alpha_d]} \frac{e^{i\mathcal{U}k\theta_{\mathfrak{a}}}}{N(\mathfrak{a})^s}$$

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Main goal is to compute $\mathrm{Var}(\psi_{K,X})$, and we use Fourier and Mellin transforms to write

$$\mathrm{Var}(\psi_{K,X}) = \frac{1}{4\pi^2 K^2} \sum_{k \neq 0} \left| \widehat{f}\left(\frac{k}{K}\right)\right|^2 \int_{(2)} \int_{(2)} \frac{L_k'}{L_k}(s) \frac{L_k'}{L_k}(\overline{s'}) \tilde{\Phi}(s) \overline{\tilde{\Phi}(s')} X^s \overline{X^{s'}} ds \overline{ds'}$$

The Ratios Conjecture

Background

- We are interested in computing the average

$$R_K(\alpha,\beta,\gamma,\delta) = \frac{1}{2K}\sum_{\substack{|k|<K\\ k\neq 0}} \frac{L_k(\frac{1}{2}+\alpha)L_k(\frac{1}{2}+\beta)}{L_k(\frac{1}{2}+\gamma)L_k(\frac{1}{2}+\delta)}.$$

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- Random Matrix Theory: Often used to model ratios of products of L-Functions, however don't capture lower order arithmetic terms.
- The Ratios conjecture is a procedure for computing averages of ratios of L-functions that predicts these lower order terms, providing a better model.

Inputs to the Ratios Conjecture

- **Approximate Functional Equation**: $L_k(s)$ can be written as

$$L_k(s) = \sum_{n<x} \frac{a_n}{n^s} + \epsilon_k(s) \sum_{m<y} \frac{\overline{a_m}}{m^{1-s}} + \text{ remainder}$$

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- **Generalized Möbius Functional Equation**: One may also write

$$\frac{1}{L_k(s)} = \sum_{h=1}^{\infty} \frac{\mu_k(h)}{h^s}$$

where μ_k is an appropriate generalization of the Möbius function.

The Procedure

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- Compute the expected value of the $\epsilon_k(s)$ factors over k, and replace them with their averages in the expanded product.
- Compute coefficient averages, and replace each component of the sum with its average.
- Extend the remaining sums out to infinity, and call the total $G(\alpha, \beta, \gamma, \delta)$. The conjecture then gives a nice expression for the average $R_K(\alpha, \beta, \gamma, \delta)$ in terms of G.

Sketch of
the Variance Calculation

Ratios Calculation

We use the ratios conjecture to analyze the expression

$$\mathrm{Var}(\psi_{K,X}) = \frac{-X}{4\pi^2K^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_{k\neq 0}\left|\widehat{f}\left(\frac{k}{K}\right)\right|^2 \frac{L_k'}{L_k}\left(\frac{1}{2}+\alpha\right)\frac{L_k'}{L_k}\left(\frac{1}{2}+\beta\right) \times \tilde{\Phi}\left(\frac{1}{2}+\alpha\right)\tilde{\Phi}\left(\frac{1}{2}+\beta\right)X^{\alpha}X^{\beta}dadb$$

where

$$\alpha = \epsilon + ia \qquad \beta = \epsilon' + ib$$

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$$\times\, \tilde{\Phi}\left(\frac{1}{2}+\alpha\right)\tilde{\Phi}\left(\frac{1}{2}+\beta\right)X^{\alpha}X^{\beta}dadb$$

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Note that

$$\frac{L_k'}{L_k}\left(\frac{1}{2}+\alpha\right)\frac{L_k'}{L_k}\left(\frac{1}{2}+\beta\right) = \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}\frac{L_k(\frac{1}{2}+\alpha)L_k(\frac{1}{2}+\beta)}{L_k(\frac{1}{2}+\gamma)L_k(\frac{1}{2}+\delta)}\bigg|_{\gamma=\alpha,\delta=\beta}$$

Ratios Calculation

From ratios conjecture, we can write

$$\sum_{|k|\leq K}\left|\widehat{f}\left(\frac{k}{K}\right)\right|^2 \frac{L_k'}{L_k}\left(\frac{1}{2}+\alpha\right)\frac{L_k'}{L_k}\left(\frac{1}{2}+\beta\right) \approx \sum_{|k|\leq K}\left|\widehat{f}\left(\frac{k}{K}\right)\right|^2 \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}R_K(\alpha,\beta,\gamma,\delta)\Big|_{\gamma=\alpha,\delta=\beta}$$

where

$$R_K(\alpha,\beta,\gamma,\delta) = \frac{1}{2K}\sum_{\substack{|k|<K\\ k\neq 0}} \frac{L_k(\frac{1}{2}+\alpha)L_k(\frac{1}{2}+\beta)}{L_k(\frac{1}{2}+\gamma)L_k(\frac{1}{2}+\delta)}$$

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- $G(\alpha, \beta, \gamma, \delta)$ = coefficient average from functional equation for $L_k(s)$
- $\langle \epsilon_k(s) \rangle_K$ = gamma factor average from functional equation for $L_k(s)$

Note: $\langle \cdot \rangle_K$ denotes average over $|k| \leq K$

Ratios Calculation

Then we have

$$\langle \epsilon_k(s) \rangle_K = \frac{1}{2-2s} \left(\frac{2\pi}{\sqrt{|D|}\frac{\mathcal{U}}{2}K} \right)^{2s-1}$$

where D is the fundamental discriminant of the number field,

i.e. $D = -4d$ or $D = -d$ depending on d mod 4

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and

$$G(\alpha, \beta, \gamma, \delta) = \frac{\zeta(1+2\alpha)\zeta(1+2\beta)\zeta(1+\alpha+\beta)\zeta(1+\gamma+\delta)}{\zeta(1+\alpha+\gamma)\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)\zeta(1+\beta+\delta)} \times \text{holomorphic function}$$

Computing the Variance

We use these averages to obtain expression for $R_K(\alpha,\beta,\gamma,\delta)$:

$$\begin{aligned} R_K(\alpha,\beta,\gamma,\delta) \approx\ & G(\alpha,\beta,\gamma,\delta) + \langle \epsilon_k(1/2+\alpha)\rangle_K G(-\alpha,\beta,\gamma,\delta) \\ & + \langle \epsilon_k(1/2+\beta)\rangle_K G(\alpha,-\beta,\gamma,\delta) \\ & + \langle \epsilon_k(1/2+\alpha)\epsilon_k(1/2+\beta)\rangle_K G(-\alpha,-\beta,\gamma,\delta) \end{aligned}$$

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and plug back into the formula for variance:

$$\begin{aligned} \mathrm{Var}(\psi_{K,X}) = \frac{-X}{4\pi^2K^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_{k\neq 0}\left|\widehat{f}\left(\frac{k}{K}\right)\right|^2 \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}R_K(\alpha,\beta,\gamma,\delta)\Big|_{\gamma=\alpha,\delta=\beta} \\ \times\,\tilde{\Phi}\left(\frac{1}{2}+\alpha\right)\tilde{\Phi}\left(\frac{1}{2}+\beta\right)X^{\alpha}X^{\beta}\,da\,db \end{aligned}$$

The Variance

Using contour integration, we arrive at our final expression for the variance:

Theorem

Assume the Ratios Conjecture. Then

$$\mathrm{Var}(\psi_{K,X}) = \begin{cases} \frac{X}{K}\left(\gamma \log K + 1\right) & \text{if } \gamma < 1 \\ \frac{X}{K}\left(\gamma \log K - 3\right) & \text{if } 1 < \gamma < 2, \\ \frac{X}{K}\left(2\log K - \log\left(\frac{\pi^2}{4}\right)\right) & \text{if } \gamma > 2 \end{cases}$$

where $X = K^\gamma$.

Conclusions

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- Variance and $G(\alpha, \beta, \gamma, \delta)$ term seemed to be independent of specific number field, as coefficient averages remained the same, and contour integration led to cancellation.
- Dependence on the specific number field appeared in the Gamma factor averages.
- Further generalizations to higher class number seem doable, but there are obstacles to this such as producing a well-defined Hecke character on non-principal ideals that still captures the notion of angle.

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Thank You!