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Playing Ball with the Largest Prime Factor An Introduction to Ruth-Aaron Numbers

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Figure: Babe Ruth

Home Run Record: 714

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Figure: Babe Ruth

Home Run Record: 714



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Figure: Babe Ruth

Home Run Record: 714



Figure: Hank Aaron

On April 8th, 1974 hit his 715th homerun

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714 and	715			

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Carl Pomerance observed some interesting facts about the numbers 714 and 715:

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714 and	715			

Carl Pomerance observed some interesting facts about the numbers 714 and 715:

- their product is the product of the first 7 primes
 - 714 * 715 = 510510 = 2 * 3 * 5 * 7 * 11 * 13 * 17
 - it is now conjectured that this is the largest pair of consecutive numbers whose product is the product of the first k primes for some k

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• the sum of the prime factors of 714 and 715 are equal

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Definition (S(n))

Suppose
$$n = p_1^{a_1} \cdots p_k^{a_k}$$
 for all p_i prime. Then define
 $S(n) = \sum_{i=1}^k a_i p_i.$

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Definition (Ruth-Aaron Number)

Suppose $n \in \mathbb{N}$ such that S(n) = S(n+1), then we call n a **Ruth-Aaron Number**.

Example

S(714)=2+3+7+17=29=5+11+13=S(715)S(77)=11+7=18=2+3+13=S(78)Thus 77 and 714 are both Ruth-Aaron Numbers

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The Gan	ne's Afoot			

In 1974 Pomerance, Carol Nelson, and David E Penney published a paper in Recreational Mathematics proving the following

Theorem

If we assume Schnizel's Hypothesis H then there are infinitely many Ruth-Aaron Numbers.

They also wrote that "The numerical data suggest that Aaron numbers are rare. We suspect they have density 0, but we cannot prove this."

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Erdős lo	ins the Team			

Erdős and Pomerance published a paper in 1978 in which they proved the first significant results regarding Ruth-Aaron Numbers.

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Theorem

The Ruth-Aaron numbers have density 0.

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Theorem

The Ruth-Aaron numbers have density 0.

Theorem

For all $\epsilon > 0$, the number of $n \leq x$ for which S(n) = S(n+1) is $O\left(\frac{x}{(\log x)^{1-\epsilon}}\right).$

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Pomerance Hits a Homerun

Shortly after Erdős's death, Pomerance proved an even stronger result:

Theorem

The number of integers $n \le x$ with S(n) = S(n+1) is $O\left(\frac{x(\log \log x)^4}{(\log x)^2}\right)$. In particular, the sum of the reciprocals of the Ruth-Aaron numbers is bounded.

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To extend these results, we consider Ruth-Aaron numbers when their prime powers have been manipulated by some nice arithmetic function and then summed.

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To extend these results, we consider Ruth-Aaron numbers when their prime powers have been manipulated by some nice arithmetic function and then summed.

Definition (K-th Power Ruth-Aaron Numbers)

Suppose $n = p_1^{a_1} \cdots p_d^{a_d}$ and we define $S_k(n) = \sum_{i=1}^d a_i p_i^k$. Then any $n \in \mathbb{N}$ such that $S_k(n) = S_k(n+1)$ then n is a k-th Power Ruth-Aaron Number.

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Definition (Euler-Totient Ruth-Aaron Numbers)

Suppose $n = p_1^{a_1} \cdots p_d^{a_d}$ and we define $f(n) = \sum_{i=1}^d a_i \varphi(p_i)$. Then any $n \in \mathbb{N}$ such that f(n) = f(n+1) is an Euler-Totient Ruth-Aaron Number.

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Theorem (Density of k-th Power Ruth-Aaron Numbers)

The K-th Power Ruth-Aaron Numbers have density 0 for all $k \in \mathbb{N}$.

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Theorem (Density of k-th Power Ruth-Aaron Numbers)

The K-th Power Ruth-Aaron Numbers have density 0 for all $k \in \mathbb{N}$.

We also prove a slightly stronger result:

Theorem

For all $\epsilon > 0$, the number of $n \leq x$ for which $S_k(n) = S_k(n+1)$ is $O(\frac{x}{\log x^{1-\epsilon}})$.

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Theorem	1			

If $n \ge 2$ is an integer, let P(n) denote the largest prime factor of n. Then we have the following theorem from Erdős and Pomerance:

Theorem (Theorem 1)

For all $\epsilon > 0$ there is a $\delta > 0$ such that for sufficiently large x, the number of $n \leq x$ with

$$\frac{1}{x^{\delta}} < \frac{P(n)}{P(n+1)} < x^{\delta}$$

is less than ϵx

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Theorem	2			

From Erdős and Pomerance we get the following Theorem for Ruth-Aaron Numbers:

Theorem (Theorem 2)

For all $\epsilon > 0$, there is a $\delta > 0$ such that for sufficiently large x there are at least $(1 - \epsilon)x$ choices for $n \leq x$ such that

 $P(n) < f(n) < (1 + x^{-\delta})P(n)$

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Theorem	2			

From Erdős and Pomerance we get the following Theorem for Ruth-Aaron Numbers:

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$$P(n) < f(n) < (1 + x^{-\delta})P(n)$$

Then we have the following analogous result for $S_k(n)$

Theorem (Theorem 2 Extended)

For all $\epsilon > 0$ there exists a $\delta > 0$ such that for sufficiently large x there are at least $(1 - \epsilon)x$ choices for $n \ge x$ such that

$$P(n)^k < S_k(n) < (1+x^{-\delta})P(n)^k$$

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Before we prove Theorem 2 we need this helpful result due to Dickman:

Theorem (Theorem A)

For every x > 0 and every $t, 0 \le t \le 1$, let A(x, t) denote the number of $n \le x$ with $P(n) \ge x^t$. Then the function

$$a(t) = \lim_{x \to \infty} x^{-1} A(x, t)$$

is defined and continuous on [0,1]

Proof of Theorem 2 (Extended)

Since any integer $n \leq x$ is divisible by at most $\frac{\log x}{\log 2}$ primes, we have for large x and composite $n \leq x$

$$S_k(n) = P(n)^k + S_k \left(\frac{n}{P(n)}\right)^k$$
$$= P(n)^k + P\left(\frac{n}{P(n)}\right)^k \frac{\log x}{\log 2}$$
$$< P(n)^k + P\left(\frac{n}{P(n)}\right)^k x^{\delta}$$

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$$< P(n)^k + P\left(\frac{n}{P(n)}\right)^k x^{\delta}$$

If Theorem 2 fails, then other than o(x) choices of $n \leq x$ we have

$$S_k(n) \ge (1+x^{-\delta})P(n)^k$$

Proof of Theorem 2 (Extended)

Thus it follows that

$$P\left(\frac{n}{P(n)}\right)^k > \frac{P(n)^k}{x^{k\delta}}$$

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Proof of Theorem 2 (Extended)

Thus it follows that

$$P\left(\frac{n}{P(n)}\right)^k > \frac{P(n)^k}{x^{k\delta}}$$

Now let $\epsilon > 0$. From Theorem A there is $\delta_0 = \delta_0(\epsilon) > 0$ such that for large x, the number of $n \leq x$ with $P(n) < x^{\delta_0}$ is at most $\frac{\epsilon x}{3}$. For each pair of primes p, q the number of $n \leq x$ with $P(n)^k = p^k$ and $P\left(\frac{n}{P(n)}\right)^k = q^k$ is at most $\left[\frac{x}{pq}\right]$.

Proof of Theorem 2 (Extended)

Hence for large x the number of $n \leq x$ for which Theorem 2 fails is at most

$$o(x) + \frac{\epsilon x}{3} + \sum_{\substack{x^{\delta_0 \leq p} \\ x^{-2\delta} p < q \leq p}} \left[\frac{x}{pq} \right] < \frac{\epsilon x}{2} + x \sum \frac{1}{p} \frac{1}{q}$$
$$< \frac{\epsilon x}{2} + \frac{4\delta x}{\delta_0},$$

if we take $\delta = \frac{\delta_0 \epsilon}{8}$, this completes the proof.

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Theorem (Theorem 1)

For all $\epsilon > 0$ there is a $\delta > 0$ such that for sufficiently large x, the number of $n \leq x$ with

$$\frac{1}{x^{\delta}} < \frac{P(n)}{P(n+1)} < x^{\delta}$$

is less than ϵx

Theorem (Theorem 2)

For all $\epsilon > 0$ there exists a $\delta > 0$ such that for sufficiently large x there are at least $(1 - \epsilon)x$ choices for $n \ge x$ such that

$$P(n)^k < S_k(n) < (1+x^{-\delta})P(n)^k$$

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Sum of Reciprocals of Euler-Totient Ruth-Aaron Numbers

Theorem

Define $f(n) = \sum_{i=1}^{d} a_i \varphi(p_i)$ for $n = a_1 p_1 \cdots a_d p_d$ where $\varphi(n)$ is the Euler-Totient function. The number of integers $n \leq x$ with f(n) = f(n+1) is $O\left(\frac{x(\log \log x)^4}{(\log x)^2}\right)$. In particular, the sum of the reciprocals of the Euler-Totient Ruth-Aaron numbers is bounded.



Similarly let P(n) denote the largest prime factor of n. Say $n \le x$ and f(n) = f(n+1). Write n = pk, n+1 = qm where p = P(n), q = P(n+1). We first note that we may assume that

$$p > x^{1/\log\log x}$$
 , $q > x^{1/\log\log x}$ (1)

since the number of integers $n \leq x$ for which (1) does not hold is

$$O\left(\frac{x}{(\log x)^2}\right).$$

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Proof of Theorem (Cont'd)

Using the fact that $\frac{t}{\log t}$ is increasing for $t \ge e$ and $\frac{2}{\log 2} < \frac{5}{\log 5}$ we get that for $P(n) \ge 5$

$$P(n) \leqslant f(N) \leqslant \frac{P(N) \log N}{\log P(N)}.$$
(2)

In light of (1), we may assume P(n), $P(n+1) \ge 5$, so that (2) holds for n and n+1.

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Proof of Theorem (Cont'd)

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We obtain the following two equations:

$$pk+1 = qm$$
, $p+f(k) = q+f(m)$

and note that the numbers k, m determine the primes p, q. Indeed,

$$p = \frac{(f(k) - f(m))m - 1}{k - m} \quad , \quad q = \frac{(f(k) - f(m))k - 1}{k - m}$$
(3)

Thus, the number of choices for *n* corresponding to choices of *k*, *m* with $k, m < \frac{x^{1/2}}{\log x}$ is at most $\frac{x}{(\log x)^2}$. We hence may assume that

$$p \leqslant x^{1/2} \log x$$
 or $q \leqslant x^{1/2} \log x$ (4)



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$$p \leqslant x^{1/2} \log x$$
 or $q \leqslant x^{1/2} \log x$ (4)

Proof of Theorem (Cont'd)

Suppose
$$p > x^{1/2} \log x$$
. Then (2) and (4) give us that $p \leqslant 2x^{1/2} \log x$

A similar inequality holds if $q > x^{1/2} \log x$. Thus we have

$$p < 2x^{1/2} \log x$$
 and $q < 2x^{1/2} \log x$ (5)

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Proof of Theorem (Cont'd)

Suppose (for now) that

$$f(k) < \frac{p}{(\log x)^2} \quad , \quad f(m) < \frac{q}{(\log x)^2} \tag{6}$$

Then we can show that

$$|p-q| < \frac{p+q}{(\log x)^2} \tag{7}$$

Now we want to count how many numbers satisfy these constraints.

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Proof of Theorem (Cont'd)

For p satisfying (1), the number of primes q such that (7) holds is $O\left(\frac{p\log\log x}{(\log x)^3}\right)$ and the sum of $\frac{1}{q}$ for such primes q is $O\left(\frac{\log\log x}{(\log x)^3}\right)$ Now, for a given choice of p, q the number of $n \le x$ with p|n and q|n+1 is at most $1 + \frac{x}{pq}$. Thus if (6) holds, the number of n that we are counting is at most

$$\sum_{p,q \text{subject to (1),(5),(7)}} 1 + \frac{x}{pq} \ll \sum_{p < 2x^{1/2} \log x} \frac{p \log \log x}{\log^3 x} + \frac{x \log \log x}{p(\log^3 x)}$$
$$\ll \frac{x \log \log x}{\log^2 x}$$

Thus we assume that (6) does not hold.

Proof of Theorem (Cont'd)

The arguments for the cases $f(k) \ge \frac{p}{(\log x)^2}$ and $f(m) \ge \frac{q}{(\log x)^2}$ are parallel, so we'll only give the details for the first case. That is, we shall assume that

$$f(k) \ge \frac{p}{(\log x)^2}.$$
(8)

First we need to establish some preliminary ideas. We write k = rl where r = P(k). Then (2) and (1) give us

$$p\frac{\log p}{2\log x} \leqslant q \leqslant p\frac{\log x}{\log p} \tag{9}$$

Additionally, (8) gives us

$$\frac{p\log p}{2(\log x)^3} \leqslant r \leqslant p \tag{10}$$

Proof of Theorem (Cont'd)

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Suppose $p \leq x^{1/3}$. Then the number of *n* in this case is at most



Thus we will assume that $p > x^{1/3}$.



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Using (3) we get the following relationship:

$$(pl - m)(rl - m) = (f(l) - f(m) - 1)ml - l + m^{2}.$$
 (11)

Thus, given l, m the number of choices of r, and hence for n, is at most

$$\tau((f(l)-f(m)-1)ml-l+m^2) \leqslant x^{o(1)},$$

where τ denotes the divisor function.

Using (3) we get the following relationship:

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Proof of Theorem (Cont'd)

If we suppose that

$$P(l) < x^{1/6}$$
 , $P(m) < x^{1/6}$ (12)

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then using some analysis we get that but for $O(x^{29/30}(\log x)^2)$ choices for $n \leq x$ we have that (12) does not hold.

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Proof of Theorem (Cont'd)

If we suppose that

$$P(l) < x^{1/6}$$
 , $P(m) < x^{1/6}$ (12)

then using some analysis we get that but for $O(x^{29/30}(\log x)^2)$ choices for $n \le x$ we have that (12) does not hold. We first consider the case that $P(I) \ge x^{1/6}$. Write I = sj where s = P(I). We rewrite (11) as

$$(psj - m)(rsj - m) = ((f(j) - f(m) - 2)mj - j)s + m^2 + mjs^2$$
(13)

We shall fix a choice for j, m and sum over choices for s.

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Lemma

Helpful Lemma

Suppose A, B, C are integers with gcd(A, B, C) = 1, $D := B^2 - 4AC \neq 0$, $A \neq 0$. Suppose the maximum value of $|At^2 + Bt + C|$ on the interval [1, x] is M_0 . Let $M = max\{M_0, |D|, x\}$, let $\mu = \lceil \frac{\log M}{\log x} \rceil$ and assume that $\mu \leqslant \frac{1}{7} \log \log x$. Then

$$\sum_{n \leqslant x} \tau(|An^2 + Bn + C|) \leqslant x(\log x)^{2^{3u+1}+4}$$

holds uniformly $x \ge x_0$. (We interpret $\tau(0)$ as 0 should it occur in the sum. The number x_0 is an absolute constant independent of the choice of A,B,C.)

Proof of Theorem (Cont'd)

We apply the lemma with A = mj, B = (f(j) - f(m) - 2)mj - jand $C = m^2$. With a little bit of work we can show that gcd(A, B, C) = 1, $D := B^2 - 4AC \neq 0$, and $A \neq 0$. Then assuming that $j < 6x^{1/6}(\log x)^2$, $m \ll x^{2/3}$, and $s \leqslant \frac{6x^{1/3}(\log x)^2}{j}$, we have that the maximum of $|As^2 + Bs + C|$ for the range of s is $\ll x^{4/3}(\log x)^2$. It follows from the lemma that

$$\sum_{s \leqslant \frac{6x^{1/3}(\log x)^2}{j}} \tau(|As^2 + Bs + C|) \leqslant \left(\frac{1}{j}\right) x^{1/3} (\log x)^c$$
(14)

for some positive constant c.

Proof of Theorem (Cont'd)

We apply the lemma with A = mj, B = (f(j) - f(m) - 2)mj - jand $C = m^2$. With a little bit of work we can show that gcd(A, B, C) = 1, $D := B^2 - 4AC \neq 0$, and $A \neq 0$. Then assuming that $j < 6x^{1/6}(\log x)^2$, $m \ll x^{2/3}$, and $s \leqslant \frac{6x^{1/3}(\log x)^2}{j}$, we have that the maximum of $|As^2 + Bs + C|$ for the range of s is $\ll x^{4/3}(\log x)^2$. It follows from the lemma that

$$\sum_{s \leq \frac{6x^{1/3}(\log x)^2}{j}} \tau(|As^2 + Bs + C|) \leq \left(\frac{1}{j}\right) x^{1/3} (\log x)^c$$
(14)

for some positive constant c.

Proof of Theorem (Cont'd)

Then if $x^{1/3} , the number of$ *n*in this case is at most

$$\sum_{p \asymp q} \left(1 + \frac{x}{pq} \right) \ll x^{2/3} (\log x)^{2c+10} + \frac{x}{\log x} \sum \frac{1}{p} \ll \frac{x \log \log x}{(\log x)^2}.$$

Thus, we may assume that $p > x^{1/3} (\log x)^{c+5}$. Then $m \ll \frac{x^{2/3}}{(\log x)^{c+5}}$, so that summing (14) over all choices for m, j we get a quantity that is $\ll \frac{x}{(\log x)^2}$.

Questions and References

Proof of Theorem (Cont'd)

Finally, we consider the remaining case when $P(m) \ge x^{1/6}$. Let m = tu where t = P(m). Then we obtain

$$(pl - tu)(rl - tu) = t^{2}(u^{2} - ul) + t(ulf(l) - ulf(u)) - l \quad (15)$$

We apply the lemma again, summing the number of divisors of the right side and get an estimate that is $\ll \frac{x}{(\log x)^{2c+2}}$, which is negligible. This completes the proof.

Introduction	Ruth-Aaron Numbers	Density	Sum of Reciprocals	Questions and References
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Open Qu	lestions			

• Is the sum of the K-th Power Ruth-Aaron Numbers bounded?

- What other arithmetic functions share these properties?
- Can this be generalized to some set of "nice" arithmetic functions?
- Can we achieve an even tighter bound on the sum?
- What can be said about triples, i.e when S(n) = S(n+1) = S(n+2), or more generally $S(n) = S(n+1) = \cdots = S(n+k)$ for some k.

Introduction 00	Ruth-Aaron Numbers 00000	Density 00000000	Sum of Reciprocals	Questions and References
Referenc	es			

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