Playing Ball with the Largest Prime Factor

An Introduction to Ruth-Aaron Numbers

Madeleine Farris

Wellesley College

July 30, 2018

The Players

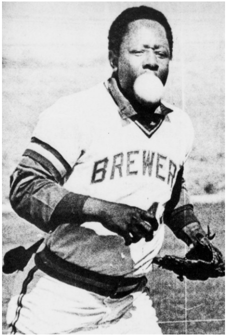
The Players

Figure: Babe Ruth

Home Run Record: 714

The Players

Figure: Babe Ruth



Home Run Record: 714

The Players

Figure: Babe Ruth

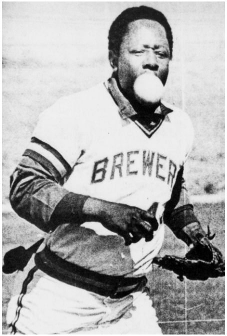


Figure: Hank Aaron

Home Run Record: 714

On April 8th, 1974 hit his 715th homerun

714 and 715

Carl Pomerance observed some interesting facts about the numbers 714 and 715:

714 and 715

Carl Pomerance observed some interesting facts about the numbers 714 and 715:

- their product is the product of the first 7 primes
  - $714 \quad 715 = 510510 = 2 \quad 3 \quad 5 \quad 7 \quad 11 \quad 13 \quad 17$
  - it is now conjectured that this is the largest pair of consecutive numbers whose product is the product of the first k primes for some k

714 and 715

Carl Pomerance observed some interesting facts about the numbers 714 and 715:

- their product is the product of the first 7 primes
  - $714 \quad 715 = 510510 = 2 \quad 3 \quad 5 \quad 7 \quad 11 \quad 13 \quad 17$
  - it is now conjectured that this is the largest pair of consecutive numbers whose product is the product of the first k primes for some k
- the sum of the prime factors of 714 and 715 are equal

Rules of the Game

Rules of the Game

Definition (S(n))

Suppose $n = p_1^{a_1} \cdots p_k^{a_k}$ *for all* p_i *prime. Then define*

$$S(n) = \sum_{i=1}^{k} a_i p_i.$$

Rules of the Game

Definition (S(n))

Suppose $n = p_1^{a_1} \cdots p_k^{a_k}$ *for all* p_i *prime. Then define*

$$S(n) = \sum_{i=1}^{k} a_i p_i.$$

Definition (Ruth-Aaron Number)

Suppose $n \in \mathbb{N}$ *such that* $S(n) = S(n+1)$, *then we call* n *a* **Ruth-Aaron Number**.

Example

S(714)=2+3+7+17=29=5+11+13=S(715)
S(77)=11+7=18=2+3+13=S(78)
Thus 77 and 714 are both Ruth-Aaron Numbers

The Game's Afoot

In 1974 Pomerance, Carol Nelson, and David E Penney published a paper in Recreational Mathematics proving the following

Theorem

If we assume Schnizel's Hypothesis H then there are in nitely many Ruth-Aaron Numbers.

They also wrote that "The numerical data suggest that Aaron numbers are rare. We suspect they have density 0, but we cannot prove this."

Erdős Joins the Team

Erdős and Pomerance published a paper in 1978 in which they proved the first significant results regarding Ruth-Aaron Numbers.

Erdős Joins the Team

Erdős and Pomerance published a paper in 1978 in which they proved the first significant results regarding Ruth-Aaron Numbers.

Theorem

The Ruth-Aaron numbers have density 0.

Erdős Joins the Team

Erdős and Pomerance published a paper in 1978 in which they proved the first significant results regarding Ruth-Aaron Numbers.

Theorem

The Ruth-Aaron numbers have density 0.

Theorem

For all $\epsilon > 0$, the number of n 6 x for which $S(n) = S(n+1)$ is $O\left(\frac{x}{(\log x)^{1\ \ \epsilon}}\right)$.

Pomerance Hits a Homerun

Shortly after Erdős's death, Pomerance proved an even stronger result:

Theorem

The number of integers n 6 x with $S(n) = S(n+1)$ is $O\left(\frac{x(\log\log x)^4}{(\log x)^2}\right)$. In particular, the sum of the reciprocals of the Ruth-Aaron numbers is bounded.

Changing the Game

Changing the Game

To extend these results, we consider Ruth-Aaron numbers when their prime powers have been manipulated by some nice arithmetic function and then summed.

Changing the Game

To extend these results, we consider Ruth-Aaron numbers when their prime powers have been manipulated by some nice arithmetic function and then summed.

Definition (K-th Power Ruth-Aaron Numbers)

Suppose $n = p_1^{a_1} \cdots p_d^{a_d}$ *and we define* $S_k(n) = \sum_{i=1}^{d} a_i p_i^k$. *Then any* $n \in \mathbb{N}$ *such that* $S_k(n) = S_k(n+1)$ *then* n *is a* k*-th Power Ruth-Aaron Number.*

Changing the Game

To extend these results, we consider Ruth-Aaron numbers when their prime powers have been manipulated by some nice arithmetic function and then summed.

Definition (K-th Power Ruth-Aaron Numbers)

Suppose $n = p_1^{a_1} \quad p_d^{a_d}$ *and we de ne* $S_k(n) = \sum_{i=1}^{d} a_i p_i^k$. *Then any* n 2 N *such that* $S_k(n) = S_k(n+1)$ *then* n *is a k-th Power Ruth-Aaron Number.*

Definition (Euler-Totient Ruth-Aaron Numbers)

Suppose $n = p_1^{a_1} \quad p_d^{a_d}$ *and we de ne* $f(n) = \sum_{i=1}^{d} a_i \varphi(p_i)$. *Then any* n 2 N *such that* $f(n) = f(n+1)$ *is an Euler-Totient Ruth-Aaron Number.*

Main Results

Theorem (Density of k-th Power Ruth-Aaron Numbers)

The K-th Power Ruth-Aaron Numbers have density 0 for all k 2 N.

Main Results

Theorem (Density of k-th Power Ruth-Aaron Numbers)

The K-th Power Ruth-Aaron Numbers have density 0 for all $k \in \mathbb{N}$.

We also prove a slightly stronger result:

Theorem

For all $\epsilon > 0$, *the number of* $n \leq x$ *for which* $S_k(n) = S_k(n+1)$ *is* $O\left(\frac{x}{\log x^{1-\epsilon}}\right)$.

Theorem 1

If $n > 2$ is an integer, let $P(n)$ denote the largest prime factor of n. Then we have the following theorem from Erdős and Pomerance:

Theorem (Theorem 1)

For all $\epsilon > 0$ there is a $\delta > 0$ such that for su ciently large x, the number of n 6 x with

$$\frac{1}{x^\delta} < \frac{P(n)}{P(n+1)} < x^\delta$$

is less than ϵx

Theorem 2

From Erdős and Pomerance we get the following Theorem for Ruth-Aaron Numbers:

Theorem (Theorem 2)

For all $\epsilon > 0$, there is a $\delta > 0$ such that for su ciently large x there are at least $(1 - \epsilon)x$ choices for $n \leqslant x$ such that

$$P(n) < f(n) < (1 + x^{-\delta})P(n)$$

Theorem 2

From Erdős and Pomerance we get the following Theorem for Ruth-Aaron Numbers:

Theorem (Theorem 2)

For all $\epsilon > 0$, there is a $\delta > 0$ such that for sufficiently large x there are at least $(1 - \epsilon)x$ choices for $n \leqslant x$ such that

$$P(n) < f(n) < (1 + x^{-\delta})P(n)$$

Then we have the following analogous result for $S_k(n)$

Theorem (Theorem 2 Extended)

For all $\epsilon > 0$ there exists a $\delta > 0$ such that for sufficiently large x there are at least $(1 - \epsilon)x$ choices for $n \geqslant x$ such that

$$P(n)^k < S_k(n) < (1 + x^{-\delta})P(n)^k$$

Before we prove Theorem 2 we need this helpful result due to Dickman:

Theorem (Theorem A)

For every $x > 0$ and every $t, 0 \le t \le 1$, let $A(x,t)$ denote the number of $n \le x$ with $P(n) > x^t$. Then the function

$$a(t) = \lim_{x \to \infty} x^{-1} A(x,t)$$

is defined and continuous on $[0,1]$

Proof of Theorem 2 (Extended)

Since any integer $n \; 6 \; x$ is divisible by at most $\frac{\log x}{\log 2}$ primes, we have for large x and composite $n \; 6 \; x$

$$
\begin{aligned}
S_k(n) &= P(n)^k + S_k\left(\frac{n}{P(n)}\right)^k \\
&= P(n)^k + P\left(\frac{n}{P(n)}\right)^k \frac{\log x}{\log 2} \\
&< P(n)^k + P\left(\frac{n}{P(n)}\right)^k x^{\delta}
\end{aligned}
$$

Proof of Theorem 2 (Extended)

Since any integer n 6 x is divisible by at most $\frac{\log x}{\log 2}$ primes, we have for large x and composite n 6 x

$$
\begin{aligned}
S_k(n) &= P(n)^k + S_k\left(\frac{n}{P(n)}\right)^k \\
&= P(n)^k + P\left(\frac{n}{P(n)}\right)^k \frac{\log x}{\log 2} \\
&< P(n)^k + P\left(\frac{n}{P(n)}\right)^k x^{\delta}
\end{aligned}
$$

If Theorem 2 fails, then other than $o(x)$ choices of n 6 x we have

$$S_k(n) > (1 + x \ ^{\delta})P(n)^k$$

Proof of Theorem 2 (Extended)

Thus it follows that

$$P\left(\frac{n}{P(n)}\right)^{k} > \frac{P(n)^{k}}{x^{k\delta}}$$

Proof of Theorem 2 (Extended)

Thus it follows that

$$P\left(\frac{n}{P(n)}\right)^k > \frac{P(n)^k}{x^{k\delta}}$$

Now let $\epsilon > 0$. From Theorem A there is $\delta_0 = \delta_0(\epsilon) > 0$ such that for large x, the number of $n \leqslant x$ with $P(n) < x^{\delta_0}$ is at most $\frac{\epsilon x}{3}$. For each pair of primes p, q the number of $n \leqslant x$ with $P(n)^k = p^k$ and $P\left(\frac{n}{P(n)}\right)^k = q^k$ is at most $\left[\frac{x}{pq}\right]$.

Proof of Theorem 2 (Extended)

Hence for large x the number of $n \leqslant x$ for which Theorem 2 fails is at most

$$o(x) + \frac{\epsilon x}{3} + \sum_{\substack{x^{\delta_0} \leqslant p \\ x \ \ {}^{2\delta} p < q \leqslant p}} \left[\frac{x}{pq} \right] < \frac{\epsilon x}{2} + x \sum \frac{1}{p} \frac{1}{q}$$

$$< \frac{\epsilon x}{2} + \frac{4\delta x}{\delta_0},$$

if we take $\delta = \frac{\delta_0 \epsilon}{8}$, this completes the proof.

Density

Theorem (Theorem 1)

For all $\epsilon > 0$ there is a $\delta > 0$ such that for su ciently large x, the number of n 6 x with

$$\frac{1}{x^{\delta}} < \frac{P(n)}{P(n+1)} < x^{\delta}$$

is less than ϵx

Theorem (Theorem 2)

For all $\epsilon > 0$ there exists a $\delta > 0$ such that for su ciently large x there are at least $(1 \quad \epsilon)x$ choices for $n > x$ such that

$$P(n)^k < S_k(n) < (1 + x^{\quad\delta})P(n)^k$$

Sum of Reciprocals of Euler-Totient Ruth-Aaron Numbers

Theorem

Define $f(n) = \sum_{i=1}^{d} a_i \varphi(p_i)$ *for* $n = a_1 p_1 \quad a_d p_d$ *where* $\varphi(n)$ *is the Euler-Totient function. The number of integers* $n \leqslant x$ *with* $f(n) = f(n+1)$ *is* $O\left(\frac{x(\log\log x)^4}{(\log x)^2}\right)$. *In particular, the sum of the reciprocals of the Euler-Totient Ruth-Aaron numbers is bounded.*

Proof of Theorem

Similarly let $P(n)$ denote the largest prime factor of n. Say $n \leqslant x$ and $f(n) = f(n+1)$. Write $n = pk$,$n + 1 = qm$ where $p = P(n)$, $q = P(n+1)$.

We first note that we may assume that

$$p > x^{1/\log\log x} \quad , \quad q > x^{1/\log\log x} \tag{1}$$

since the number of integers $n \leqslant x$ for which (1) does not hold is

$$O\left(\frac{x}{(\log x)^2}\right).$$

Proof of Theorem (Cont'd)

Using the fact that $\frac{t}{\log t}$ is increasing for $t > e$ and $\frac{2}{\log 2} < \frac{5}{\log 5}$ we get that for $P(n) > 5$

$$P(n) \; 6 \; f(N) \; 6 \; \frac{P(N) \log N}{\log P(N)}. \tag{2}$$

In light of (1), we may assume $P(n), P(n+1) > 5$, so that (2) holds for n and $n+1$.

Proof of Theorem (Cont'd)

Using the fact that $\frac{t}{\log t}$ is increasing for $t > e$ and $\frac{2}{\log 2} < \frac{5}{\log 5}$ we get that for $P(n) > 5$

$$P(n)\ 6\ f(N)\ 6\ \frac{P(N)\log N}{\log P(N)}. \qquad (2)$$

In light of (1), we may assume $P(n), P(n+1) > 5$, so that (2) holds for n and $n+1$.

Proof of Theorem (Cont'd)

We obtain the following two equations:

$$pk + 1 = qm \quad , \quad p + f(k) = q + f(m)$$

and note that the numbers k, m determine the primes p, q. Indeed,

$$p = \frac{(f(k) \quad f(m))m \quad 1}{k \quad m} \quad , \quad q = \frac{(f(k) \quad f(m))k \quad 1}{k \quad m} \qquad (3)$$

Thus, the number of choices for n corresponding to choices of k, m with $k, m < \frac{x^{1/2}}{\log x}$ is at most $\frac{x}{(\log x)^2}$. We hence may assume that

$$p \ 6 \ x^{1/2} \log x \quad \text{or} \quad q \ 6 \ x^{1/2} \log x \qquad (4)$$

Proof of Theorem (Cont'd)

We obtain the following two equations:

$$pk + 1 = qm \quad , \quad p + f(k) = q + f(m)$$

and note that the numbers k, m determine the primes p, q. Indeed,

$$p = \frac{(f(k) \quad f(m))m \quad 1}{k \quad m} \quad , \quad q = \frac{(f(k) \quad f(m))k \quad 1}{k \quad m} \qquad (3)$$

Thus, the number of choices for n corresponding to choices of k, m with $k, m < \frac{x^{1/2}}{\log x}$ is at most $\frac{x}{(\log x)^2}$. We hence may assume that

$$p \; 6 \; x^{1/2} \log x \quad \text{or} \quad q \; 6 \; x^{1/2} \log x \qquad (4)$$

Proof of Theorem (Cont'd)

We obtain the following two equations:

$$pk + 1 = qm \quad , \quad p + f(k) = q + f(m)$$

and note that the numbers k, m determine the primes p, q. Indeed,

$$p = \frac{(f(k) \quad f(m))m \quad 1}{k \quad m} \quad , \quad q = \frac{(f(k) \quad f(m))k \quad 1}{k \quad m} \qquad (3)$$

Thus, the number of choices for n corresponding to choices of k, m with $k, m < \frac{x^{1/2}}{\log x}$ is at most $\frac{x}{(\log x)^2}$. We hence may assume that

$$p \ 6 \ x^{1/2} \log x \quad \text{or} \quad q \ 6 \ x^{1/2} \log x \qquad (4)$$

Proof of Theorem (Cont'd)

We obtain the following two equations:

$$pk + 1 = qm \quad , \quad p + f(k) = q + f(m)$$

and note that the numbers k, m determine the primes p, q. Indeed,

$$p = \frac{(f(k) \quad f(m))m \quad 1}{k \quad m} \quad , \quad q = \frac{(f(k) \quad f(m))k \quad 1}{k \quad m} \qquad (3)$$

Thus, the number of choices for n corresponding to choices of k, m with $k, m < \frac{x^{1/2}}{\log x}$ is at most $\frac{x}{(\log x)^2}$. We hence may assume that

$$p \;\; 6 \;\; x^{1/2} \log x \quad \text{or} \quad q \;\; 6 \;\; x^{1/2} \log x \qquad (4)$$

Proof of Theorem (Cont'd)

Suppose $p > x^{1/2}\log x$. Then (2) and (4) give us that

$$p \leqslant 2x^{1/2}\log x$$

A similar inequality holds if $q > x^{1/2}\log x$. Thus we have

$$p < 2x^{1/2}\log x \quad \text{and} \quad q < 2x^{1/2}\log x \tag{5}$$

Proof of Theorem (Cont'd)

Suppose (for now) that

$$f(k) < \frac{p}{(\log x)^2} \quad , \quad f(m) < \frac{q}{(\log x)^2} \tag{6}$$

Then we can show that

$$jp \quad qj < \frac{p+q}{(\log x)^2} \tag{7}$$

Now we want to count how many numbers satisfy these constraints.

Proof of Theorem (Cont'd)

For p satisfying (1), the number of primes q such that (7) holds is $O\left(\frac{p \log\log x}{(\log x)^3}\right)$ and the sum of $\frac{1}{q}$ for such primes q is $O\left(\frac{\log\log x}{(\log x)^3}\right)$ Now, for a given choice of p, q the number of n 6 x with pjn and $qjn + 1$ is at most $1 + \frac{x}{pq}$. Thus if (6) holds, the number of n that we are counting is at most

$$\sum_{p,q \text{subject to } (1),(5),(7)} 1 + \frac{x}{pq} \qquad \sum_{p<2x^{1/2}\log x} \frac{p \log\log x}{\log^3 x} + \frac{x \log\log x}{p(\log^3 x)}$$

$$\frac{x \log\log x}{\log^2 x}$$

Thus we assume that (6) does not hold.

Proof of Theorem (Cont'd)

The arguments for the cases $f(k) > \frac{p}{(\log x)^2}$ and $f(m) > \frac{q}{(\log x)^2}$ are parallel, so we'll only give the details for the first case. That is, we shall assume that

$$f(k) > \frac{p}{(\log x)^2}. \tag{8}$$

First we need to establish some preliminary ideas. We write $k = rl$ where $r = P(k)$. Then (2) and (1) give us

$$p\frac{\log p}{2\log x} \leqslant q \leqslant p\frac{\log x}{\log p} \tag{9}$$

Additionally, (8) gives us

$$\frac{p\log p}{2(\log x)^3} \leqslant r \leqslant p \tag{10}$$

Proof of Theorem (Cont'd)

The arguments for the cases $f(k) > \frac{p}{(\log x)^2}$ and $f(m) > \frac{q}{(\log x)^2}$ are parallel, so we'll only give the details for the first case. That is, we shall assume that

$$f(k) > \frac{p}{(\log x)^2}. \tag{8}$$

First we need to establish some preliminary ideas. We write $k = rl$ where $r = P(k)$. Then (2) and (1) give us

$$p\frac{\log p}{2\log x} \ 6 \ q \ 6 \ p\frac{\log x}{\log p} \tag{9}$$

Additionally, (8) gives us

$$\frac{p\log p}{2(\log x)^3} \ 6 \ r \ 6 \ p \tag{10}$$

Proof of Theorem (Cont'd)

The arguments for the cases $f(k) > \frac{p}{(\log x)^2}$ and $f(m) > \frac{q}{(\log x)^2}$ are parallel, so we'll only give the details for the first case. That is, we shall assume that

$$f(k) > \frac{p}{(\log x)^2}. \tag{8}$$

First we need to establish some preliminary ideas. We write $k = rl$ where $r = P(k)$. Then (2) and (1) give us

$$p\frac{\log p}{2\log x} \leqslant q \leqslant p\frac{\log x}{\log p} \tag{9}$$

Additionally, (8) gives us

$$\frac{p\log p}{2(\log x)^3} \leqslant r \leqslant p \tag{10}$$

Proof of Theorem (Cont'd)

Suppose $p \leqslant x^{1/3}$. Then the number of n in this case is at most

$$\sum_{\substack{p,q,r \text{ subject to } (2.1),(2.8),(2.9), \\ p \leqslant x^{1/3}}} 1 + \frac{x}{prq}$$

$$\frac{x}{\log^3 x} + \sum_{p > x^{1/\log\log x}} \frac{x}{p} \frac{\log\log x}{\log p} \frac{\log\log x}{\log p}$$

$$\frac{x(\log\log x)^4}{(\log x)^2}.$$

Thus we will assume that $p > x^{1/3}$.

Proof of Theorem (Cont'd)

Suppose $p \leqslant x^{1/3}$. Then the number of n in this case is at most

$$\sum_{\substack{p,q,r \text{ subject to } (2.1),(2.8),(2.9), \\ p \leqslant x^{1/3}}} 1 + \frac{x}{prq}$$

$$\frac{x}{\log^3 x} + \sum_{p > x^{1/\log\log x}} \frac{x}{p} \frac{\log\log x}{\log p} \frac{\log\log x}{\log p}$$

$$\frac{x(\log\log x)^4}{(\log x)^2}.$$

Thus we will assume that $p > x^{1/3}$.

Proof of Theorem (Cont'd)

Using (3) we get the following relationship:

$$(pl \quad m)(rl \quad m) = (f(l) \quad f(m) \quad 1)ml \quad l + m^2. \qquad (11)$$

Thus, given l, m the number of choices of r, and hence for n, is at most

$$\tau((f(l) \quad f(m) \quad 1)ml \quad l + m^2) \; 6 \; x^{o(1)},$$

where τ denotes the divisor function.

Proof of Theorem (Cont'd)

Using (3) we get the following relationship:

$$(pl \quad m)(rl \quad m) = (f(l) \quad f(m) \quad 1)ml \quad l + m^2. \qquad (11)$$

Thus, given l, m the number of choices of r, and hence for n, is at most

$$\tau((f(l) \quad f(m) \quad 1)ml \quad l + m^2) \; 6 \; x^{o(1)},$$

where τ denotes the divisor function.

Proof of Theorem (Cont'd)

If we suppose that

$$P(l) < x^{1/6} \quad , \quad P(m) < x^{1/6} \tag{12}$$

then using some analysis we get that but for $O(x^{29/30}(\log x)^2)$ choices for n 6 x we have that (12) does not hold.

Proof of Theorem (Cont'd)

If we suppose that

$$P(l) < x^{1/6} \quad , \quad P(m) < x^{1/6} \tag{12}$$

then using some analysis we get that but for $O(x^{29/30}(\log x)^2)$ choices for $n \; 6 \; x$ we have that (12) does not hold.
We first consider the case that $P(l) > x^{1/6}$. Write $l = sj$ where $s = P(l)$. We rewrite (11) as

$$(psj \quad m)(rsj \quad m) = ((f(j) \quad f(m) \quad 2)mj \quad j)s + m^2 + mjs^2 \tag{13}$$

We shall fix a choice for j, m and sum over choices for s.

Helpful Lemma

Lemma

Suppose A, B, C are integers with $\gcd(A, B, C) = 1$, $D := B^2 - 4AC \neq 0$, $A \neq 0$. Suppose the maximum value of $|At^2 + Bt + C|$ on the interval $[1, x]$ is M_0. Let $M = \max\{M_0, |D|, x\}$, let $\mu = \lceil \frac{\log M}{\log x} \rceil$ and assume that $\mu \leq \frac{1}{7} \log\log x$. Then

$$\sum_{n \leq x} \tau(|An^2 + Bn + C|) \leq x (\log x)^{2^{3u+1}+4}$$

holds uniformly $x > x_0$. (We interpret $\tau(0)$ as 0 should it occur in the sum. The number x_0 is an absolute constant independent of the choice of A,B,C.)

Proof of Theorem (Cont'd)

We apply the lemma with $A = mj$, $B = (f(j) \quad f(m) \quad 2)mj \quad j$ and $C = m^2$. With a little bit of work we can show that $\gcd(A, B, C) = 1$, $D := B^2 \quad 4AC \neq 0$, and $A \neq 0$. Then assuming that $j < 6x^{1/6}(\log x)^2, m \quad x^{2/3}$, and $s \leq \frac{6x^{1/3}(\log x)^2}{j}$, we have that the maximum of $jAs^2 + Bs + Cj$ for the range of s is $x^{4/3}(\log x)^2$. It follows from the lemma that

$$\sum_{s \leq \frac{6x^{1/3}(\log x)^2}{j}} \tau(jAs^2 + Bs + Cj) \leq \left(\frac{1}{j}\right) x^{1/3}(\log x)^c \qquad (14)$$

for some positive constant c.

Proof of Theorem (Cont'd)

We apply the lemma with $A = mj$, $B = (f(j) \quad f(m) \quad 2)mj \quad j$ and $C = m^2$. With a little bit of work we can show that $\gcd(A, B, C) = 1$, $D := B^2 \quad 4AC \; 6= 0$, and $A \; 6= 0$. Then assuming that $j < 6x^{1/6}(\log x)^2, m \quad x^{2/3}$, and $s \; 6 \; \frac{6x^{1/3}(\log x)^2}{j}$, we have that the maximum of $jAs^2 + Bs + Cj$ for the range of s is $x^{4/3}(\log x)^2$. It follows from the lemma that

$$\sum_{s \, 6 \, \frac{6x^{1/3}(\log x)^2}{j}} \tau(jAs^2 + Bs + Cj) \; 6 \; \left(\frac{1}{j}\right) x^{1/3}(\log x)^c \qquad (14)$$

for some positive constant c.

Proof of Theorem (Cont'd)

Then if $x^{1/3} < p \; 6 \; x^{1/3}(\log x)^{c+5}$, the number of n in this case is at most

$$\sum_{p \quad q} \left(1 + \frac{x}{pq}\right) \qquad x^{2/3}(\log x)^{2c+10} + \frac{x}{\log x} \sum \frac{1}{p} \qquad \frac{x \log \log x}{(\log x)^2}.$$

Thus, we may assume that $p > x^{1/3}(\log x)^{c+5}$. Then $m \qquad \frac{x^{2/3}}{(\log x)^{c+5}}$, so that summing (14) over all choices for m, j we get a quantity that is $\qquad \frac{x}{(\log x)^2}$.

Proof of Theorem (Cont'd)

Finally, we consider the remaining case when $P(m) > x^{1/6}$. Let $m = tu$ where $t = P(m)$. Then we obtain

$$(pl \quad tu)(rl \quad tu) = t^2(u^2 \quad ul) + t(ulf(l) \quad ulf(u)) \quad l \qquad (15)$$

We apply the lemma again, summing the number of divisors of the right side and get an estimate that is $\quad \frac{x}{(\log x)^{2c+2}}$, which is negligible. This completes the proof.

Open Questions

- Is the sum of the K-th Power Ruth-Aaron Numbers bounded?
- What other arithmetic functions share these properties?
- Can this be generalized to some set of "nice" arithmetic functions?
- Can we achieve an even tighter bound on the sum?
- What can be said about triples, i.e when $S(n) = S(n+1) = S(n+2)$, or more generally $S(n) = S(n+1) = \quad = S(n+k)$ for some k.

References

1. P. Erdős and C. Pomerance, *On the largest prime factors of n and $n+1$*, Aequationes Mathematicae, **17** (1978), 311-321.
2. C. Pomerance, *Ruth-Aaron Numbers Revisited*, Paul Erdős and his Mathematics, **I** (2002), 567-579.