Explicit Sato-Tate for Primes in Arithmetic Progressions

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Background

Historical prime number theorems:

$$\pi(x) := \#\{p \le x \mid p \text{ is prime}\}$$
$$\pi(x) \sim \mathrm{Li(x)} := \int_2^x \frac{\mathrm{dt}}{\log \mathrm{t}}$$
$$\pi(x;q,a) := \#\{p \le x \mid p \equiv a \bmod q\}$$
$$\pi(x;q,a) \sim \frac{1}{\varphi(q)}\mathrm{Li(x)}$$

1.) Introduction

Let

$$f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k^{\text{new}}(\Gamma_0(N)),\ q = e^{2\pi i z}$$

be a weight k, level N newform. It follows from Deligne's proof of the Weil conjectures that $a_f(p) \le 2p^{(k-1)/2}$ and hence we can write

$$a_f(p) = 2p^{(k-1)/2}\cos\theta_p.$$

The distribution of these θ_p is given by the Sato-Tate Conjecture, which is now a theorem due to Barnet-Lamb, Geraghy, Harris, and Taylor [1]. Let

$$\pi_{f,I}(x) := \#\{p \le x : \theta_p \in I\}.$$

Then

$$\pi_{f,I}(x) \sim \mu_{\mathrm{ST}}\mathrm{Li(x)},$$

where $d\mu_{\mathrm{ST}} = \frac{2}{\pi}\sin^2\theta\, d\theta$ is the Sato-Tate measure.

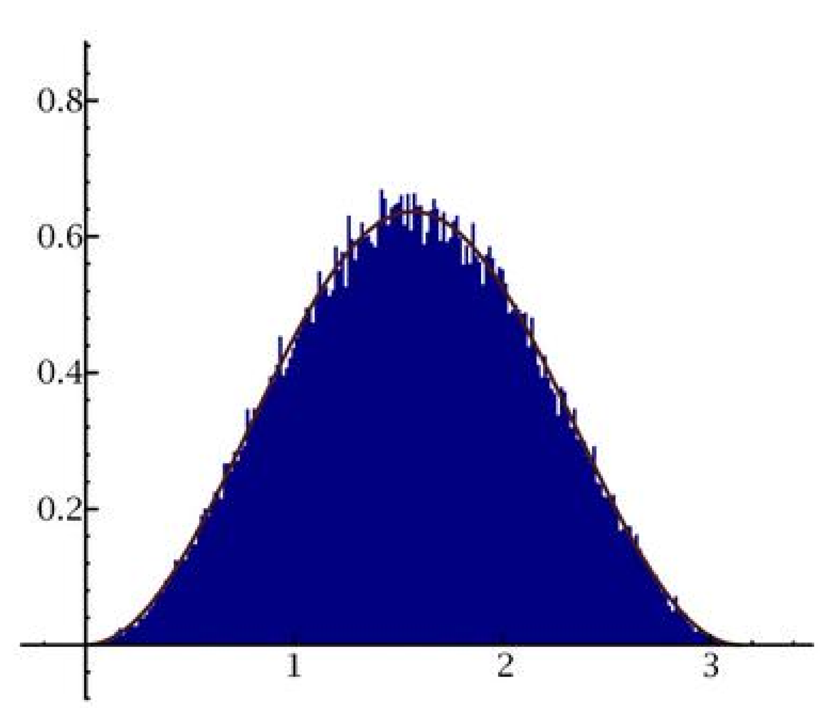

Figure 1: The horizontal axis is the interval $0 \le \theta \le \pi$ and the height of a subinterval is proportional to the percentage of primes $p < 10^6$ such that θ_p lies in the given subinterval. [2]

2.) Previous Work

In the work of Jeremy Rouse and Jesse Thorner [3], they prove an explicit error term for the Sato-Tate conjecture, namely, for a newform f with level N and weight k, they obtain

$$|\#\{p \le x : \theta_p \in I\} - \mu_{\mathrm{ST}}\mathrm{Li(x)}| \ll \frac{x^{3/4}\log(Nkx)}{\log x},$$

with an implied constant of 3.33, for sufficiently large x. This result assumes the Generalized Riemann Hypothesis and certain analytic hypotheses of symmetric power L-functions. They also improve best known upper bounds for the Lang-Trotter conjecture assuming GRH.

4.) Other Results

We may also study the distribution of primes for which $a_f(p) = c$ for a fixed constant $c \in \mathbb{R}$. Using Deligne's bound, we have

$$a_f(p) = 2p^{(k-1)/2}\cos\theta_p = c,$$

and as p grows, θ_p must approach $\pi/2$. Studying the primes for which θ_p is close to $\pi/2$ allows us to produce bounds on the quantity

$$\pi_{c,f}(x) := \#\{x < p \le 2x : a_f(p) = c\}.$$

Let $f(z) = \sum_{n=1}^{\infty}\tau(n)q^n$, where $\tau(n)$ is the Ramanujan tau function. We prove for all $x \ge 10^{40}$,

$$\pi_{0,f}(x) \le 6.03 \times 10^{-7}\frac{x^{3/4}}{\sqrt{\log x}}.$$

Main Result: Sato-Tate Conjecture for Primes in Arithmetic Progressions

Assume the Generalized Riemann Hypothesis for the twisted symmetric power L-functions of f, and that $\gcd(a,q) = 1$. We compute constants $C_1 = C_1(\phi, f)$ and $C_2 = C_2(\phi)$ such that if $x > C_1$ then

$$\left|\sum_{\substack{p \equiv a(q) \\ \theta_p \in I}} \log(p)\phi(p/x) - \frac{x}{\varphi(q)}\mu_{ST}(I)\left(\int_{-\infty}^{\infty}\phi(t)\,dt\right)\right| \le \left(2.54\frac{\sqrt{q}}{\varphi(q)}\int_{-\infty}^{\infty}\phi(t)\,dt + 1.24\frac{C_2}{\sqrt{q}}\right)x^{3/4}\sqrt{\log x}$$

3.) Setup

Instead of using the exact indicator function for the interval $[1,2]$, we use a smooth, compactly supported test function ϕ that is a pointwise upper bound for the indicator function. Analogously to how the Riemann zeta function is used to prove the prime number theorem, we consider symmetric power L-functions in the same way, and twist them by a Dirichlet character in order to restrict to an arithmetic progression, giving a refinement of Sato-Tate. Additionally, we weigh each prime by its logarithm, which makes the analysis significantly more tractable.

5.) Lehmer's Conjecture

Serre [4] proved that if $f(z)$ is a newform, then

$$\lim_{x\to\infty}\frac{\#\{n \le x : a(n) \ne 0\}}{x} = \alpha_f \prod_{a(p)=0}\left(1 - \frac{1}{p+1}\right).$$

Lehmer conjectured that if $f(z) = \sum_{n=1}^{\infty}\tau(n)q^n$,then

$$\lim_{x\to\infty}\frac{\#\{n \le x : \tau(n) \ne 0\}}{x} = \prod_{\tau(p)=0}\left(1 - \frac{1}{p+1}\right) = 1.$$

We prove that

$$\lim_{x\to\infty}\frac{\#\{n \le x : \tau(n) \ne 0\}}{x} > 1 - 5.101 \times 10^{-14}.$$

6.) Further Work

With current techniques, it seems that a resolution to Lehmer's conjecture is not yet possible. Even improving the bound for the density remains difficult due optimization issues in intermediate ranges. Using the main result, one could produce bounds for $\pi_{0,f}(x)$ for other newforms f.

7.) References

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Acknowledgements

This research was supervised by Steven J. Miller and Jesse Thorner at the Williams College SMALL REU and was supported by the National Science Foundation (grant number DMS-1659037). The second named presenter was also supported by Princeton University. The presenters used *Mathematica* 11.3 for explicit calculations.

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