When Bands Play in Random Matrix Theory

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SMALL Undergraduate Research

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Bands?

Random Matrix Theory?

$$\begin{bmatrix} 9 & 13 & 5 & 2 \\ 1 & 11 & 7 & 6 \\ 3 & 7 & 4 & 1 \\ 6 & 0 & 7 & 10 \end{bmatrix}$$

Objectives

- Review classical random matrix theory.
- Introduce band matrices
- Investigate the dependence of the limiting eigenvalue distribution on the number of bands.

Classical Random Matrix Theory

Origins of Random Matrix Theory

Nuclear physics (1950's): Solve for energy eigenfunctions.

Fundamental Equation:

$$H\psi_n = E_n\psi_n$$

H : matrix, entries depend on system

E_n : energy levels

ψ_n : energy eigenfunctions

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Strategy

- Average over eigenvalues of random matrices.
- Hope system behaves close to average.

Random Matrix Example: Real Symmetric

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,N} \\ a_{1,2} & a_{2,2} & a_{2,3} & \cdots & a_{2,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1,N} & a_{2,N} & a_{3,N} & \cdots & a_{N,N} \end{pmatrix} = A^T$$

Fix p, define

$$\text{Prob}(A) = \prod_{1 \leq i \leq j \leq N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \leq i \leq j \leq N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

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Want to understand eigenvalues of A.

Eigenvalue Density Measure

$$\mu_{A,N}(x)dx \;=\; \frac{1}{N}\sum_{i=1}^{N}\delta\left(x-\frac{\lambda_i(A)}{\sqrt{N}}\right)dx.$$

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Let

$$M_k \;=\; \lim_{N\to\infty}\mathbb{E}_A\left[M_k(A,N)\right];$$

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Want to understand the eigenvalues of A, but **we cannot compute the eigenvalues directly**.

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Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\mathrm{Trace}(A^k) \;=\; \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\mathrm{Trace}(A^k) \;=\; \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1,i_2} a_{i_2,i_3} \cdots a_{i_k,i_1}.$$

Wigner's Semi-Circle Law

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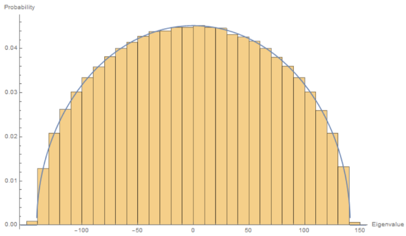
$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost every A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

New limiting distributions

Different behavior emerges as symmetry increases!

From semi-circle to ...



Full palindromic Toeplitz

[MMS] Massey, Miller and Sinsheimer (2007)

For real symmetric palindromic matrices, converge in probability to the Gaussian (if p is even, then have strong convergence)

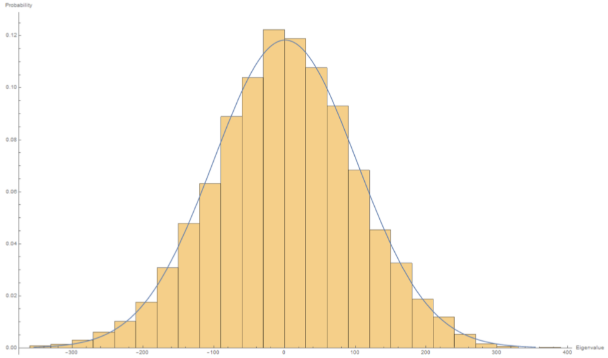


Figure 1: Eigenvalue distribution Symmetric Palindromic Toeplitz $10K$x$10K$

Banded Matrices

Why banded matrices?

Banded matrices have a long history; **band one matrices** (not constant on diagonals) are related to the **Laplacian of some systems in mathematical physics**.

Example of a Band One Matrix:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & 0 & 0 & \cdots & 0 \\ a_{1,2} & a_{2,2} & a_{2,3} & 0 & \cdots & 0 \\ 0 & a_{2,3} & a_{3,3} & a_{2,4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & a_{N-1,N} \\ 0 & 0 & 0 & \cdot\cdot & a_{N-1,N} & a_{N,N} \end{pmatrix}$$

Our Work

We study a new case: $N \times N$ **banded symmetric palindromic Toeplitz (BSPT) matrices** constructed by adding constant diagonals to the center and corners of the zero matrix.

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Example: 8-by-8 real symmetric palindromic Toeplitz matrix with 2 bands:

$$\begin{pmatrix} 0 & a_1 & a_2 & 0 & 0 & a_2 & a_1 & 0 \\ a_1 & 0 & a_1 & a_2 & 0 & 0 & a_2 & a_1 \\ a_2 & a_1 & 0 & a_1 & a_2 & 0 & 0 & a_2 \\ 0 & a_2 & a_1 & 0 & a_1 & a_2 & 0 & 0 \\ 0 & 0 & a_2 & a_1 & 0 & a_1 & a_2 & 0 \\ a_2 & 0 & 0 & a_2 & a_1 & 0 & a_1 & a_2 \\ a_1 & a_2 & 0 & 0 & a_2 & a_1 & 0 & a_1 \\ 0 & a_1 & a_2 & 0 & 0 & a_2 & a_1 & 0 \end{pmatrix}.$$

Our Work

The imposed **additional symmetry of our palindromic Toeplitz case greatly increases the number of terms that contribute** to the moments of the eigenvalue distribution compared to the case of band matrices which are not constant along diagonals.

Approach

Entries are drawn from **probability distribution p with mean 0, variance 1, and finite higher moments**.

We calculate the moments using **method of moments**. Due to the banded nature of our matrices, this work turns out different than previous published work:

We look at the *expected value* for the moments:

$$M_n(N) \quad := \quad \mathbb{E}(M_n(A,N)) = \frac{1}{N(2D)^{\frac{k}{2}}} \sum_{1 \le i_1, \ldots, i_n \le N} \mathbb{E}(a_{i_1,i_2} a_{i_2,i_3} \cdots a_{i_n,i_1}).$$

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For small values of D, our matrices are sparse and few terms contribute in the moment calculation as $N \to \infty$. If D is comparable to $\frac{N}{2}$, however, the same behavior as the full palindromic Toeplitz emerges.

Approach (Cont'd)

[DMMM] Devlin, Ma, Mattos da Silva, and Miller (2019)

If we fix a number of bands D and let the size of matrix N tend to infinity, only the terms in the bulk of the matrix contribute to the moment.

Note that for each row in the bulk, we have $2D$ choices of columns for which the chosen entry is nonzero.

Results

FIRST DIFFICULTY: Normalization

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Recall the $2k^{\text{th}}$ moment of a Gaussian distribution is $(2k-1)!!$

Second Moment

Let p_k denote the moments of the initial distribution, e.g. p_2 is the second moment of p.
For the second moment calculation, we consider

$$\mathbb{E}[a_{i,j}a_{j,i}] = \mathbb{E}[a_{i,j}^2]$$

We have $(N - 2D)$ rows in the bulk with $2D$ choices of non-zero entries

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Since we have attached a point mass of size $\frac{1}{N}$ at each eigenvalue, we must normalize the quantity

$$2D - 2\frac{D}{N}.$$

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If we wish for the second moment to be 1 in the limit $N \to \infty$, we must normalize it by a factor of $\sqrt{2D}$.
Hence, the moment equation becomes

$$M_k(A, N, D) = \frac{1}{N}\frac{1}{(2D)^{\frac{k}{2}}}\sum_{i=1}^{N}\lambda_i^k(A)$$

Fourth Moment

We consider $\mathbb{E}[a_{i,j}a_{j,k}a_{k,l}a_{l,i}]$.

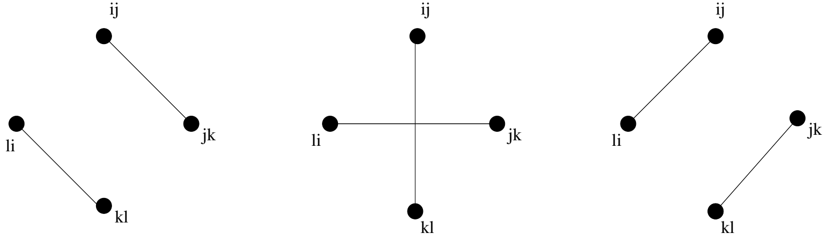

Figure 2: Adjacent pairings and non-adjacent pairing configuration.

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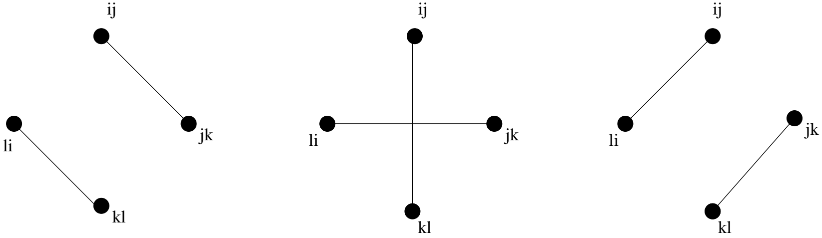

Figure 2: Adjacent pairings and non-adjacent pairing configuration.

In contrast to [MMS], a quadruple pairing will contribute to our fourth moment since our normalization factor differs from $\frac{1}{N^3}$.

Fourth Moment

Adjacent case: $x_1 = x_2$ and $x_3 = x_4$:

$$j - i \;=\; -(k - j) + 0 \qquad\qquad \ell - k \;=\; -(i - \ell) + 0,$$

which implies

$$i \;=\; k$$

and ℓ is arbitrary.

This gives that the contribution is

$$(N - 2D)(2D)(2D - 2)p_2^2$$

where p_2 is second moment of the probability distribution p.

Fourth Moment

Non-adjacent case: $x_1 = x_3$ and $x_2 = x_4$:

$$j - i \;=\; -(\ell - k) + C_1 \qquad\qquad k - j \;=\; -(i - \ell) + C_2,$$

or equivalently

$$j \;=\; i + k - \ell + C_1 \;=\; i + k - \ell - C_2.$$

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No Diophantine obstruction! (Similar to [MMS].)

It follows that the contribution is

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Fourth Moment

Quadruple case: $x_1 = x_3 = x_2 = x_4$:

Using the same counting approach, we see that

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With the adjacent and non-adjacent cases the fourth moment is

$$M_4 = \left(3 - \frac{3}{D}\right)p_2^2 + \left(\frac{1}{2D}\right)p_4.$$

Higher Moments

Any higher moment can be calculated by brute force computation and more complicated combinatorics.

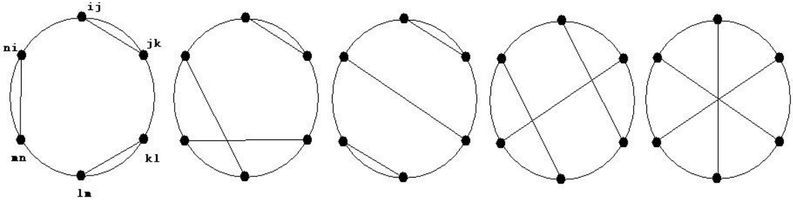

Figure 3: Configurations for the six moment

$$M_6 = 15\left(1 - \frac{3}{D} + \frac{2}{D^2}\right)p_2^3 + 15\left(\frac{1}{2D} - \frac{1}{2D^2}\right)p_2 p_4 + \left(\frac{1}{4D^2}\right)p_6$$

where p_2, p_4, p_6 are second, fourth and sixth moments of p (respectively).

Distributions for different D's

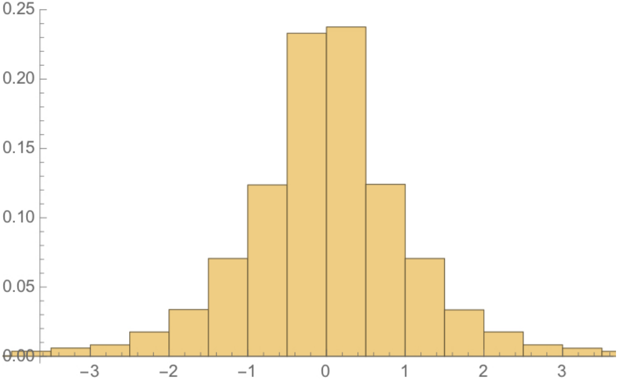

Figure 4: Distribution for 1 band, size 200×200.

Initial distribution p has mean 0, variance 1, and finite higher moments.

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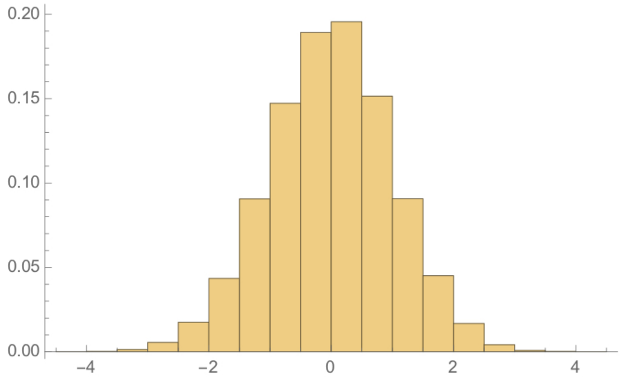

Figure 5: Distribution for 50 bands, size 200×200.

Initial distribution p has mean 0, variance 1, and finite higher moments.

Lower Bound: Unbounded Support

Lemma: Suppose $D \geq 2$. If p has mean 0 and variance 1, then

$$M_{2k,D} \geq (2k-1)!! \cdot F(k,D)$$

where $F(k,D) > 0$ is a constant for fixed D.

Conjecture: $F(k,D)$ is bounded below uniformly in k by a fixed positive constant.

Corollary: Devlin, Ma, Mattos da Silva, and Miller (2019)

$$\lim_{k\to\infty} \sqrt[2k]{M_{2k,D}} = \infty$$

Hence, the support of the limiting spectral measure is unbounded.

Upper Bound: Unique Distribution

Carleman's Condition

Let μ be a measure on $\mathbb{R}$ which has finite moments of all orders. Then μ is uniquely determined by its moments provided that

$$\sum_{k=1}^{\infty} M_{2k}^{-1/(2k)} = \infty.$$

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Assume p is a distribution with mean 0, variance 1, and finite higher moments. Need moments of p to grow slow enough that the sum of reciprocals of the $2k^{\text{th}}$ root of our upper bound diverges.

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Conjecture: For $k \geq 4$ and

$$M_{2k,D} \leq \mathcal{P}(k) \cdot (2k-1)!! \left(\sum_{i=0}^{k-1} (-1)^i s(k, i+1) \left(\frac{1}{D} \right)^i \right) p_{2k} \quad (1)$$

where $\mathcal{P}(k)$ is the number of integer partitions of a set with k elements.

Future Work

- Investigate strength of upper bound, and determine if moments of eigenvalue distribution determine a unique distribution.
- Determine a closed form for the higher even moments (calculate 8^{th} moment explicitly).
- Deduce further convergence results.

Acknowledgements

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Thank you for your attention!