Crescent Configurations Under Non-Euclidean Norms

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Outline

- Erdős distinct distances problem
- Crescent configurations under Euclidean norms
- Crescent configurations under $L^p$ norms
  - Line-like configurations in $L^p$
  - Crescent configurations in $L^p$
Erdős distinct distances problem

Question [Erdős, 1946]

Given $n$ points in a plane, what is the minimum number of distinct distances $\Delta(n)$ that they determine?

We “expect” $\binom{n}{2} = O(n^2)$ distinct distances. How low can we go?
Erdős Distinct Distances Problem: Bounds

Upper bounds:
- \( \Delta(n) = O\left(\frac{n}{\sqrt{\log n}}\right) \)

Lower bounds:
- \( \Delta(n) = \Omega(n^{1/2}) \) (Erdős, 1946)
- \( \Delta(n) = \Omega(n^{2/3}) \) (Moser, 1952)
- \( \Delta(n) = \Omega(n^{5/7}) \) (Chung, 1984)
- \( \Delta(n) = \Omega(n^{4/5}/\log n) \) (Chung + Szemerédi + Trotter, 1992)
- \( \Delta(n) = \Omega(n^{4/5}) \) (Székely, 1993)
- \( \Delta(n) = \Omega(n^{6/7}) \) (Solymosi + Tóth, 2001)
- \( \Delta(n) = \Omega(n^{4\epsilon/5\epsilon-1}) \approx \Omega(n^{0.8636}) \) (Tardos, 2003)
- \( \Delta(n) = \Omega(n^{48-14\epsilon/55-16\epsilon}) \approx \Omega(n^{0.8641}) \) (Katz + Tardos, 2004)
- \( \Delta(n) = \Omega(n^{\frac{n}{\log n}}) \) (Guth + Katz, 2015)
Erdős Distinct Distances Problem: Variants

- The structure of all near-optimal point sets (which obtain $O\left(\frac{n}{\sqrt{\log n}}\right)$)
- Restriction: no 3 points on a line
- Restriction: no 3 points on a line and no 4 points on a circle (general position)
- Restriction: points must be in convex position
- Higher (and lower) dimensions
- Bipartite problems (points lie on one of two lines)
- Distinct distances with local properties
- Crescent configurations
Erdős’ Question

Question [Erdős, 1989]

Does there exist a set of $n$ points such that:

1. The $n$ points determine $n - 1$ distinct distances
2. For all $1 \leq i \leq n - 1$, there exists a distance which occurs exactly $i$ times

Answer: Yes!

1. $n$ equally spaced points on a line
2. $n$ equally spaced points on a circular arc
Erdős’ Crescent configurations

To rule out these trivial configurations, Erdős introduced an additional requirement that the points lie in general position.

**Definition**

We say that \( n \) points in the plane lie in **general position** if no three points lie on a common line and no four points lie on a common circle.

This leads to the definition of a crescent configuration.

**Definition**

We say that \( n \) points in the plane form a **crescent configuration** if:

1. The \( n \) points lie in general position
2. The \( n \) points determine \( n - 1 \) distinct distances
3. For all \( 1 \leq i \leq n - 1 \), there exists a distance which occurs exactly \( i \) times
Current results about crescent configurations

For $3 \leq n \leq 8$, constructions are known (Erdős, I. Pálásti, A. Liu, and C. Pomerance).

For $n \geq 9$, it is an open problem whether crescent configurations of size $n$ exist.
Crescent configurations are rare: heuristics

We “expect” crescent configurations to be extremely rare.

**Definition**

We say that $n$ points in the plane form a **crescent configuration** if:

1. The $n$ points lie in general position
2. The $n$ points determine $n - 1$ distinct distances
3. For all $1 \leq i \leq n - 1$, there exists a distance which occurs exactly $i$ times

- By Guth and Katz (2015), $n$ points determine $\Omega\left(\frac{n}{\log n}\right)$ distinct distances. Just $n - 1$ distinct distances is cutting close!
- The general position condition is very restrictive.
- The multiplicity condition is very restrictive.
We examine how crescent configurations behave under a generalization of the $L^2$ norm, the $L^p$ norm.

**Definition ($L^p$ distance)**

Let $1 \leq p < \infty$. Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two points in the plane. Their $L^p$ distance is given by:

$$d_p(a, b) = (|b_x - a_x|^p + |b_y - a_y|^p)^{1/p}$$

There is also the notion of the $L^\infty$ norm.

**Definition ($L^\infty$ distance)**

Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two points in the plane. Their $L^\infty$ distance is given by:

$$d_p(a, b) = \max\{|b_x - a_x|, |b_y - a_y|\}$$
**$L^p$ unit balls and perpendicular bisectors**

**Unit ball**: set of points which have 1 from the origin.

**Perpendicular bisector**: set of points which are equidistant from two given points.

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<tr>
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<th>$L^1$</th>
<th>$L^2$</th>
<th>$L^3$</th>
<th>$L^\infty$</th>
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**Legend**
- Background
- $L^2$ crescent configs
- $L^p$ Geometry
- $L^p$ line-like configs
- $L^p$ crescent configs
- End
Crescent configurations in $L^p$

Now we can ask the same question about crescent configurations in $L^p$.

Question [in $L^p$]

Does there exist a set of $n$ points such that:

1. The $n$ points determine $n - 1$ distinct distances
2. For all $1 \leq i \leq n - 1$, there exists a distance which occurs exactly $i$ times

Recall in $L^2$, we introduced the condition that the points must lie in general position in order to eliminate trivial crescent configurations.

Step 1: For each $1 \leq p \leq \infty$, find all trivial crescent configurations in $L^p$.

Step 2: Introduce a condition in the definition of $L^p$ crescent configurations to eliminate these trivial configurations.
Line-like configurations

Recall the trivial crescent configurations in $L^2$:

Key observation: The distance graphs of all of these trivial crescent configurations are isomorphic to the distance graph of $n$ equally spaced points on a line.

Definition

We say that $n$ points in the plane form a **line-like configuration** if their distance graph is isomorphic to the distance graph of $n$ equally spaced points on a line.
Definition

We say that \( n \) points in the plane form a **line-like configuration** if their distance graph is isomorphic to the distance graph of \( n \) equally spaced points on a line.

The trivial crescent configurations in \( L^p \) are precisely the line-like configurations.

Definition (\( L^p \) crescent configuration)

We say that \( n \) points in the plane form a **crescent configuration** if:

1. The \( n \) points do not contain a line-like configuration of size four
2. The \( n \) points determine \( n - 1 \) distinct distances
3. For all \( 1 \leq i \leq n - 1 \), there exists a distance which occurs exactly \( i \) times
Constructing line-like configurations: A geometrical approach

For $1 \leq p \leq \infty$, we can construct line-like configurations in $L^p$ using the same general approach.
**Conjecture**

For $p \in (1, \infty) \setminus \{2\}$, the only line-like configurations of size $n > 4$ are sets of equally spaced points on a line.

Reasoning: We have numerical evidence (Mathematica) which suggests that no other line-like configurations exist. Trying to geometrically construct a line-like configuration which does not lie on a straight line results in near-misses:
We have a large family of $L^1$ line-like configurations, for example:

We can show that all $L^1$ line-like configurations are of this form by a geometrical argument:
There are four types of line-like configurations in $L^\infty$.

- **straight**: $n \geq 3$
- **screw**: $3 \leq n \leq 4$
- **staircase**: $3 \leq n \leq 6$
- **twisted**: ?

**Conjecture**

For sufficiently large $n$, every $L^\infty$ line-like configuration is straight.
Our results show that:

1. Line-like configurations have four different types of behavior for $p = 1$, $p = 2$, $p \in (1, \infty) \setminus \{2\}$, and $p = \infty$.

2. Having an understanding of the line-like configurations in $L^p$ means that we have an understanding of the trivial crescent configurations in $L^p$. 
Crescent configurations in $L^p$

**Definition ($L^p$ crescent configuration)**

We say that $n$ points in the plane form a **crescent configuration** if:

1. The $n$ points do not contain a line-like configuration of size four
2. The $n$ points determine $n - 1$ distinct distances
3. For all $1 \leq i \leq n - 1$, there exists a distance which occurs exactly $i$ times

Recall: Crescent configurations are rare. In $L^2$, it is an open problem whether crescent configurations of size $n$ exist for $n \geq 9$.

**Our Question**

In $L^p$, for which $n$ do there exist crescent configurations of size $n$?
Crescent configurations in $L^p$, $1 < p < \infty$

We have a construction for a crescent configuration of size $n = 4$. 
Crescent configurations in $L^1$

We have constructions for crescent configurations in $L^1$ of size $3 \leq n \leq 4$.

$P_1 = (0, 0)$
$P_2 = \left(\frac{3}{2}, \frac{1}{2}\right)$
$P_3 = \left(\frac{3}{2}, -\frac{1}{2}\right)$
$P_4 = (-2, 0)$
Crescent configurations in $L^\infty$

We have constructions for crescent configurations in $L^\infty$ of size $3 \leq n \leq 7$.

\[ P_1 = (0, 0) \]
\[ P_2 = (2, 1) \]
\[ P_3 = (1, 3) \]
\[ P_4 = (4, -1) \]
\[ P_5 = (1, -2) \]
\[ P_6 = (5, -3) \]
\[ P_7 = (-1, -4) \]
Future Work

Continuations of our work

- Rigorously understanding line-like configurations in $L^p$, $1 < p < \infty$
- Constructing more crescent configurations in $L^p$, $1 \leq p \leq \infty$

Extensions of our work

- Applying our $L^p$ framework to other discrete geometry problems (other than the problem of crescent configurations)
- Searching for $L^2$ crescent configurations in higher dimensional spaces
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Questions

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