Crescent Configurations Under Non-Euclidean Norms

Students: Sara Fish, Dylan King, Catherine Wahlenmayer
Mentors: Prof. Steven J. Miller, Prof. Eyvindur Palsson

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Outline

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Crescent configurations under Euclidean norms

Crescent configurations under L^p norms

- Line-like configurations in L^p
- Crescent configurations in L^p

Erdős distinct distances problem

Question [Erdős, 1946]

Given n points in a plane, what is the minimum number of distinct distances (n) that they determine?

We "expect" $\binom{n}{2} = O(n^2)$ distinct distances. How low can we go?

Erd}os Distinct Distances Problem: Bounds

Upper bounds:

$(\ n) = O(p\frac{n}{\log n})$

Lower bounds:

$(\ n) = (\ n^{1=2})$ (Erd}os, 1946)

$(\ n) = (\ n^{2=3})$ (Moser, 1952)

$(\ n) = (\ n^{5=7})$ (Chung, 1984)

$(\ n) = (\ n^{4=5}=\log n)$ (Chung + Szemeredi + Trotter, 1992)

$(\ n) = (\ n^{4=5})$ (Szekely, 1993)

$(\ n) = (\ n^{6=7})$ (Solymosi + Toth, 2001)

$(\ n) = (\ n^{\frac{4}{5\ \ 1}})\quad (\ n^{0:8636})$ (Tardos, 2003)

$(\ n) = (\ n^{\frac{48\ \ 14}{55\ \ 16}})\quad (\ n^{0:8641})$ (Katz + Tardos, 2004)

$(\ n) = (\ \frac{n}{\log n})$ (Guth + Katz, 2015)

Erdős Distinct Distances Problem: Variants

The structure of all near-optimal point sets (which obtain $O(\frac{n}{\sqrt{\log n}})$)

Restriction: no 3 points on a line

Restriction: no 3 points on a line and no 4 points on a circle (general position)

Restriction: points must be in convex position

Higher (and lower) dimensions

Bipartite problems (points lie on one of two lines)

Distinct distances with local properties

Crescent con gurations

Erdős' Question

Question [Erdős, 1989]

Does there exist a set of n points such that:

1. The n points determine $n-1$ distinct distances
2. For all $1 \le i \le n-1$, there exists a distance which occurs exactly i times

Answer: Yes!

- n equally spaced points on a line
- n equally spaced points on a circular arc

Erdős' Crescent configurations

To rule out these trivial configurations, Erdős introduced an additional requirement that the points lie in general position.

Definition

We say that n points in the plane lie in *general position* if no three points lie on a common line and no four points lie on a common circle.

This leads to the definition of a crescent configuration.

Definition

We say that n points in the plane form a *crescent configuration* if:

1. The n points lie in general position
2. The n points determine $n - 1$ distinct distances
3. For all $1 \le i \le n - 1$, there exists a distance which occurs exactly i times

Current results about crescent configurations

For $3 \le n \le 8$, constructions are known (Erdős, I. Palásti, A. Liu, and C. Pomerance).

For $n \ge 9$, it is an open problem whether crescent configurations of size n exist.

Crescent con gurations are rare: heuristics

We \expect" crescent con gurations to be extremely rare.

De nition

We say that n points in the plane form a crescent con guration if:

1. The n points lie in general position
2. The n points determine n 1 distinct distances
3. For all 1 i n 1, there exists a distance which occurs exactly i times

By Guth and Katz (2015), n points determine $(\frac{n}{\log n})$ distinct distances. Just n 1 distinct distances is cutting close!

The general position condition is very restrictive.

The multiplicity condition is very restrictive.

L^p norm

We examine how crescent configurations behave under a generalization of the L^2 norm, the L^p norm.

Definition (L^p distance)

Let $1 \le p < \infty$. Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two points in the plane. Their L^p distance is given by:

$$d_p(a,b) = (|b_x - a_x|^p + |b_y - a_y|^p)^{1/p}$$

There is also the notion of the L^∞ norm.

Definition (L^∞ distance)

Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two points in the plane. Their L^∞ distance is given by:

$$d_p(a,b) = \max\{|b_x - a_x|, |b_y - a_y|\}$$

L^p unit balls and perpendicular bisectors

Unit ball : set of points which have 1 from the origin.

Perpendicular bisector : set of points which are equidistant from two given points.

L^1	L^2	L^3	L^1

Crescent configurations in L^p

Now we can ask the same question about crescent configurations in L^p.

Question [in L^p]

Does there exist a set of n points such that:

1. The n points determine $n-1$ distinct distances
2. For all $1 \le i \le n-1$, there exists a distance which occurs exactly i times

Recall in L^2, we introduced the condition that the points must lie in general position in order to eliminate trivial crescent configurations.

Step 1: For each $1 \le p \le \infty$, find all trivial crescent configurations in L^p.
Step 2: Introduce a condition in the definition of L^p crescent configurations to eliminate these trivial configurations.

Line-like configurations

Recall the trivial crescent configurations in L^2:

Key observation: The distance graphs of all of these trivial crescent configurations are isomorphic to the distance graph of n equally spaced points on a line.

Definition

We say that n points in the plane form a *line-like configuration* if their distance graph is isomorphic to the distance graph of n equally spaced points on a line.

L^p crescent con gurations

De nition

We say that n points in the plane form a line-like con guration if their distance graph is isomorphic to the distance graph of n equally spaced points on a line.

The trivial crescent con gurations in L^p are precisely the line-like con gurations.

De nition (L^p crescent con guration)

We say that n points in the plane form a crescent con guration if:

1. The n points do not contain a line-like con guration of size four
2. The n points determine $n \quad 1$ distinct distances
3. For all $1 \quad i \quad n \quad 1$, there exists a distance which occurs exactly i times

Constructing line-like configurations: A geometrical approach

For $1 \le p \le \infty$, we can construct line-like configurations in L^p using the same general approach.

L^p line-like configurations, $p \in (1;\infty) \setminus \{2\}$

Conjecture

For $p \in (1;\infty) \setminus \{2\}$, the only line-like configurations of size $n > 4$ are sets of equally spaced points on a line.

Reasoning: We have numerical evidence (Mathematica) which suggest that no other line-like configurations exist. Trying to geometrically construct a line-like configuration which does not lie on a straight line results in near-misses:

L^1 line-like con gurations

We have a large family of L^1 line-like con gurations, for example

We can show that all L^1 line-like con gurations are of this form by a geometrical argument:

L^1 line-like con gurations

There are four types of line-like con gurations in L^1 .

straight	screw	staircase	twisted
n 3	3 n 4	3 n 6	?

Conjecture

For su ciently large n, every L^1 line-like con guration is straight.

Line-like configurations: summary

Our results show that:

- Line-like configurations have four different types of behavior for $p = 1$, $p = 2$, $p \in (1;\infty) \setminus \{2\}$, and $p = \infty$.
- Having an understanding of the line-like configurations in L^p means that we have an understanding of the trivial crescent configurations in L^p.

Crescent con gurations in L^p

De nition (L^p crescent con guration)

We say that n points in the plane form a crescent con guration if:

1. The n points do not contain a line-like con guration of size four
2. The n points determine n 1 distinct distances
3. For all 1 i n 1, there exists a distance which occurs exactly i times

Recall: Crescent con gurations are rare. In L^2, it is an open problem whether crescent con gurations of size n exist for n 9.

Our Question

In L^p, for which n do there exist crescent con gurations of size n?

Crescent con gurations in L^p, $1 < p < 1$

We have a construction for a crescent con guration of size $n = 4$.

Crescent con gurations in L^1

We have constructions for crescent con gurations in L^1 of size $3 \quad n \quad 4$.

$$P_1 = (0\,;0)$$
$$P_2 = (\tfrac{3}{2}; \tfrac{1}{2})$$
$$P_3 = (\tfrac{3}{2}; \quad \tfrac{1}{2})$$
$$P_4 = (\quad 2; 0)$$

Crescent con gurations in L^1

We have constructions for crescent con gurations in L^1 of size $3 \quad n \quad 7$.

$P_1 = (0\,;0)$

$P_2 = (2\,;1)$

$P_3 = (1\,;3)$

$P_4 = (4\,; \quad 1)$

$P_5 = (1\,; \quad 2)$

$P_6 = (5\,; \quad 3)$

$P_7 = (\quad 1; \quad 4)$

Future Work

Continuations of our work

- Rigorously understanding line-like con gurations in L^p, $1 < p < 1$
- Constructing more crescent con gurations in L^p, $1 \quad p \quad 1$

Extensions of our work

- Applying our L^p framework to other discrete geometry problems (other than the problem of crescent con gurations)
- Searching for L^2 crescent con gurations in higher dimensional space

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