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Crescent Configurations Under Non-Euclidean Norms

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SMALL REU at Williams College

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Outline					

- Erdős distinct distances problem
- Crescent configurations under Euclidean norms
- Crescent configurations under L^p norms
 - Line-like configurations in L^p
 - Crescent configurations in L^p

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Erdős distinct distances problem

Question [Erdős, 1946]

Given *n* points in a plane, what is the minimum number of distinct distances $\Delta(n)$ that they determine?

We "expect" $\binom{n}{2} = O(n^2)$ distinct distances. How low can we go?



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Erdős Distinct Distances Problem: Bounds

Upper bounds:

• $\Delta(n) = O(\frac{n}{\sqrt{\log n}})$

Lower bounds:

- $\Delta(n) = \Omega(n^{1/2})$ (Erdős, 1946)
- $\Delta(n) = \Omega(n^{2/3})$ (Moser, 1952)
- $\Delta(n) = \Omega(n^{5/7})$ (Chung, 1984)
- $\Delta(n) = \Omega(n^{4/5}/\log n)$ (Chung + Szemerédi + Trotter, 1992)
- $\Delta(n) = \Omega(n^{4/5})$ (Székely, 1993)
- $\Delta(n) = \Omega(n^{6/7})$ (Solymosi + Tóth, 2001)
- $\Delta(n) = \Omega(n^{\frac{4\epsilon}{5\epsilon-1}}) \approx \Omega(n^{0.8636})$ (Tardos, 2003)
- $\Delta(n) = \Omega(n^{\frac{48-14\epsilon}{55-16\epsilon}}) \approx \Omega(n^{0.8641})$ (Katz + Tardos, 2004)
- $\Delta(n) = \Omega(\frac{n}{\log n})$ (Guth + Katz, 2015)



Erdős Distinct Distances Problem: Variants

- The structure of all near-optimal point sets (which obtain $O(\frac{n}{\sqrt{\log n}})$)
- Restriction: no 3 points on a line
- Restriction: no 3 points on a line and no 4 points on a circle (general position)
- Restriction: points must be in convex position
- Higher (and lower) dimensions
- Bipartite problems (points lie on one of two lines)
- Distinct distances with local properties
- Crescent configurations



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Erdős' Question

Question [Erdős, 1989]

Does there exist a set of n points such that:

- **①** The *n* points determine n 1 distinct distances
- ② For all 1 ≤ i ≤ n − 1, there exists a distance which occurs exactly i times

Answer: Yes!

- In equally spaced points on a line
- In equally spaced points on a circular arc



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Erdős' Crescent configurations

To rule out these trivial configurations, Erdős introduced an additional requirement that the points lie in general position.

Definition

We say that *n* points in the plane lie in **general position** if no three points lie on a common line and no four points lie on a common circle.

This leads to the definition of a crescent configuration.

Definition

We say that n points in the plane form a **crescent configuration** if:

- **1** The *n* points lie in general position
- **2** The *n* points determine n 1 distinct distances
- So For all 1 ≤ i ≤ n − 1, there exists a distance which occurs exactly i times

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Current results about crescent configurations

For $3 \le n \le 8$, constructions are known (Erdős, I. Pàlàsti, A. Liu, and C. Pomerance).



For $n \ge 9$, it is an open problem whether crescent configurations of size n exist.

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Crescent configurations are rare: heuristics

We "expect" crescent configurations to be extremely rare.

Definition

We say that n points in the plane form a crescent configuration if:

- The *n* points lie in general position
- 2 The *n* points determine n-1 distinct distances
- So For all 1 ≤ i ≤ n − 1, there exists a distance which occurs exactly i times
 - By Guth and Katz (2015), n points determine Ω(n log n) distinct distances. Just n 1 distinct distances is cutting close!
 - The general position condition is very restrictive.
 - The multiplicity condition is very restrictive.

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L ^p norm					

We examine how crescent configurations behave under a generalization of the L^2 norm, the L^p norm.

Definition (L^p distance)

Let $1 \le p < \infty$. Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two points in the plane. Their L^p distance is given by:

$$d_p(a,b) = (|b_x - a_x|^p + |b_y - a_y|^p)^{1/p}$$

There is also the notion of the L^{∞} norm.

Definition (L^{∞} distance)

Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two points in the plane. Their L^{∞} distance is given by:

$$d_p(a, b) = \max\{|b_x - a_x|, |b_y - a_y|\}$$

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L^p unit balls and perpendicular bisectors

Unit ball: set of points which have 1 from the origin.

Perpendicular bisector: set of points which are equidistant from two given points.



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Crescent configurations in L^p

Now we can ask the same question about crescent configurations in L^{p} .

Question [in L^p]

Does there exist a set of n points such that:

- **①** The *n* points determine n-1 distinct distances
- ② For all 1 ≤ i ≤ n − 1, there exists a distance which occurs exactly i times

Recall in L^2 , we introduced the condition that the points must lie in general position in order to eliminate trivial crescent configurations.

Step 1: For each $1 \le p \le \infty$, find all trivial crescent configurations in L^p .

Step 2: Introduce a condition in the definition of L^p crescent configurations to eliminate these trivial configurations.

Background	L ² crescent configs	L ^p Geometry	L ^p line-like configs	L ^p crescent configs	End
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Line-like configurations

Recall the trivial crescent configurations in L^2 :



Key observation: The distance graphs of all of these trivial crescent configurations are isomorphic to the distance graph of n equally spaced points on a line.

Definition

We say that n points in the plane form a **line-like configuration** if their distance graph is isomorphic to the distance graph of n equally spaced points on a line.

Background	L ² crescent configs	L ^p Geometry	L ^p line-like configs	L ^p crescent configs	End
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L ^p cresc	ent configuratio	ons			

Definition

We say that n points in the plane form a **line-like configuration** if their distance graph is isomorphic to the distance graph of n equally spaced points on a line.

The trivial crescent configurations in L^p are precisely the line-like configurations.

Definition (*L^p* crescent configuration)

We say that n points in the plane form a **crescent configuration** if:

- The *n* points do not contain a line-like configuration of size four
- ② The *n* points determine n 1 distinct distances
- So For all 1 ≤ i ≤ n − 1, there exists a distance which occurs exactly i times



Constructing line-like configurations: A geometrical approach

For $1 \le p \le \infty$, we can construct line-like configurations in L^p using the same general approach.



Background	L ² crescent configs	L ^p Geometry	L ^p line-like configs	L ^p crescent configs	End
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L^p line-like configurations, $p \in (1,\infty) \setminus \{2\}$

Conjecture

For $p \in (1, \infty) \setminus \{2\}$, the only line-like configurations of size n > 4 are sets of equally spaced points on a line.

Reasoning: We have numerical evidence (Mathematica) which suggests that no other line-like configurations exist. Trying to geometrically construct a line-like configuration which does not lie on a straight line results in near-misses:



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L¹ line-like configurations

We have a large family of L^1 line-like configurations, for example



We can show that all L^1 line-like configurations are of this form by a geometrical argument:





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L^{∞} line-like configurations

There are four types of line-like configurations in L^{∞} .



Conjecture

For sufficiently large n, every L^{∞} line-like configuration is straight.

Background	L ² crescent configs	L ^p Geometry	L ^p line-like configs	L ^p crescent configs	End
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Line-like configurations: summary

Our results show that:

- Line-like configurations have four different types of behavior for $p = 1, p = 2, p \in (1, \infty) \setminus \{2\}$, and $p = \infty$.
- Having an understanding of the line-like configurations in L^p means that we have an understanding of the trivial crescent configurations in L^p.

Background	L ² crescent configs	L ^p Geometry	L ^p line-like configs	L ^p crescent configs	End
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Crescent configurations in L^p

Definition (L^p crescent configuration)

We say that n points in the plane form a crescent configuration if:

- **1** The *n* points do not contain a line-like configuration of size four
- 2 The *n* points determine n-1 distinct distances
- So For all 1 ≤ i ≤ n − 1, there exists a distance which occurs exactly i times

Recall: Crescent configurations are rare. In L^2 , it is an open problem whether crescent configurations of size n exist for $n \ge 9$.

Our Question

In L^p , for which *n* do there exist crescent configurations of size *n*?

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Crescent configurations in L^p , 1

We have a construction for a crescent configuration of size n = 4.



Background	L ² crescent configs	L ^p Geometry	L ^p line-like configs	L ^p crescent configs	End
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Crescent configurations in L^1

We have constructions for crescent configurations in L^1 of size $3 \le n \le 4$.



$$P_{1} = (0,0)$$

$$P_{2} = (\frac{3}{2}, \frac{1}{2})$$

$$P_{3} = (\frac{3}{2}, -\frac{1}{2})$$

$$P_{4} = (-2,0)$$



Crescent configurations in L^{∞}

We have constructions for crescent configurations in L^{∞} of size $3 \le n \le 7$.



$$P_1 = (0,0)$$

$$P_2 = (2,1)$$

$$P_3 = (1,3)$$

$$P_4 = (4,-1)$$

$$P_5 = (1,-2)$$

$$P_6 = (5,-3)$$

$$P_7 = (-1,-4)$$

Future Wo	ork				
Background	L ² crescent configs	L ^p Geometry	L ^p line-like configs	L ^p crescent configs	End
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Continuations of our work

- Rigorously understanding line-like configurations in $L^p,\, 1$
- Constructing more crescent configurations in L^p , $1 \leq p \leq \infty$

Extensions of our work

- Applying our *L^p* framework to other discrete geometry problems (other than the problem of crescent configurations)
- Searching for L^2 crescent configurations in higher dimensional spaces

Acknowla	decomonto				
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Questions



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