

Previous Results

The Fibonacci Numbers: Let $F_1 = 1, F_2 = 2$ and for $n \geq 2$

$$F_n = F_{n-1} + F_{n-2}.$$

Zeckendorf's Theorem

Every positive integer can be written uniquely as the sum of non-consecutive Fibonacci numbers where

$$F_n = F_{n-1} + F_{n-2}$$

and $F_1 = 1, F_2 = 2$.

The Fibonacci Quilt Sequence



Figure 1: The first 19 terms of the Fibonacci Quilt Sequence, constructed on the Log Cabin quilt pattern. Starting with 1 in the center and for each subsequent term adding the smallest integer that cannot be expressed as the sum of non-adjacent previous terms.

Given an increasing sequence of positive integers $\{q_i\}_{i=1}^{\infty}$. A decomposition of an integer

$$m = q_{l_1} + q_{l_2} + \cdots + q_{l_t}$$

(where $q_{l_i} > q_{l_{i+1}}$) is an *FQ-legal decomposition*¹ if for all $i, j, |l_i - l_j| \neq 0, 1, 3, 4$ and $\{1, 3\} \not\subset$ $\{l_1, l_2, \ldots, l_t\}.$

The Fibonacci Quilt Game

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Definition¹

The Fibonacci Quilt Sequence is an increasing sequence of positive integers $\{q_i\}_{i=1}^{\infty}$, where every $q_i \ (i \geq 1)$ is the smallest positive integer that does not have an FQ-legal decomposition using the elements $\{q_1, ..., q_{i-1}\}$.

Theorem¹

Let q_n denote the n^{th} term in the Fibonacci Quilt, then

> for $n \ge 5$, $q_{n+1} = q_{n-1} + q_{n-2}$, for $n \ge 6$, $q_{n+1} = q_n + q_{n-4}$.

The Game

The Fibonacci Quilt Game was inspired by the Zeckendorf $Game^2$. It is a two player game, beginning with k 1's, where players alternate applying the following rules until no further moves can be made. The winner is the player who makes the last move.

Rule 1: For
$$n \ge 2$$
, $q_n + q_{n+1} \to q_{n+3}$
Base Rule: $1 + 2 \to 3$
Rule 2: For $n \ge 2$, $q_n + q_{n+4} \to q_{n+5}$
Base Rule: $1 + 5 \to 4 + 2 \bigstar$
Rule 3: For $n \ge 7$, $2q_n \to q_{n+2} + q_{n-5}$
Base Rules: $1 + 1 \to 2$, $2 + 2 \to 4$,
 $3 + 3 \to 2 + 4$, $5 + 5 \to 1 + 9$
 $4 + 4 \to 3 + 5$ or $4 + 4 \to 1 + 7 \bigstar$
 $7 + 7 \to 5 + 9$ or $7 + 7 \to 2 + 12 \bigstar$
Rule 4: For $n \ge 7$, $q_n + q_{n+3} \to q_{n-5} + q_{n+4}$
Base Rules: $1 + 4 \to 5$, $2 + 5 \to 7$,
 $3 + 7 \to 1 + 9$, $4 + 9 \to 1 + 12$
 $5 + 12 \to 1 + 16$, $7 + 16 \to 2 + 21$

Rule 5: $1 + 3 \rightarrow 4$ \star This rule may only be applied when no other moves are possible.

 \star The player may decide which move to make.

On Figure 1 each illegal distance from 4 is colored. Each rule corrects a specific illegal pair of terms in the current decomposition. For example, Rule would take $4 + 5 \rightarrow 9$.

Theorem Every game terminates in a finite number of moves at an FQ-legal decomposition.

To prove this we use a monovarient, the sum of the square roots of the indices of the terms. Each turn decreases this since the following

Example Game on n = 8

We will use the notation that $1^6 \wedge 2^3$ means 6 1's and 3 2's. Start with 1^8 . Player 1 applies Rule 3: $1^6 \wedge 2$. Player 2 applies Rule 1: $1^5 \wedge 3$. Player 1 applies Rule 5: $1^4 \wedge 4$. Player 2 applies Rule 4: $1^3 \wedge 5$. Player 1 applies Rule 3: $1 \land 2 \land 5$. Player 2 applies Rule 4: $1 \wedge 7$. Player 2 wins.

The Game is Playable

- Rule 1: $\sqrt{n+3} \sqrt{n} \sqrt{n+1}$
- Rule 2: $\sqrt{n+5} \sqrt{n} \sqrt{n+4}$
- Rule 3: $\sqrt{n+2} + \sqrt{n-5} 2\sqrt{n}$ Rule 4: $\sqrt{n+4} + \sqrt{n-5} - \sqrt{n} - \sqrt{n+3}$
- are all negative, as well as the base rules, with one exception: $1+5 \rightarrow 2+4 \bigstar$, thus we restrict the use of this rule.

Lower Bound on Game Length

Notation: Let L(n) denote the maximum number of terms in an FQ-legal decomposition of n. Let l(n)denote the minimum number of terms in an FQ-legal decomposition of n.

Example: 50 = 1 + 49 = 2 + 4 + 16 + 28L(50) = 4, l(50) = 2.

Theorem

The shortest possible game on n is completed in n - L(n) moves.

Distribution of Game Length



deviation.

	Moment	Random Game	Gaussian	Percent Difference
	2	11653.9	11653.9	0
	4	1.36314×10^{8}	1.36311×10^8	2.5253×10^{-3}
	6	1.60017×10^{12}	1.59997×10^{12}	0.0128609
Table 1: The moments of our distribution compared to the				
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Gaussian curve with the same mean and standard deviation calculated in Mathematica.

1 M. Catral, P.L. Ford, P.E. Harris, S.J. Miller, D. Nelson, Legal Decomposition Arising From Non-Positive Linear Recurrences. Fibonacci Quarterly (54 (2016), no. 4, 348-365). 2 P. Baird-Smith, A. Epstein, K. Flint, S.J. Miller, The Zeckendorf Game. (2018).

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Conjecture

As n goes to infinity, the number of moves in a random game, when all legal moves are equally likely, converges to a Gaussian.



Figure 2: The distribution of 5000 random games on n = 100, and the Gaussian curve with the same mean and standard

References

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