The Fibonacci Quilt Game

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Previous Results

The Fibonacci Numbers: Let $F_1 = 1$, $F_2 = 2$ and for $n >= 2$

$$F_n = F_{n-1} + F_{n-2}.$$

Zeckendorf's Theorem

Every positive integer can be written uniquely as the sum of non-consecutive Fibonacci numbers where

$$F_n = F_{n-1} + F_{n-2}$$

and $F_1 = 1$, $F_2 = 2$.

The Fibonacci Quilt Sequence

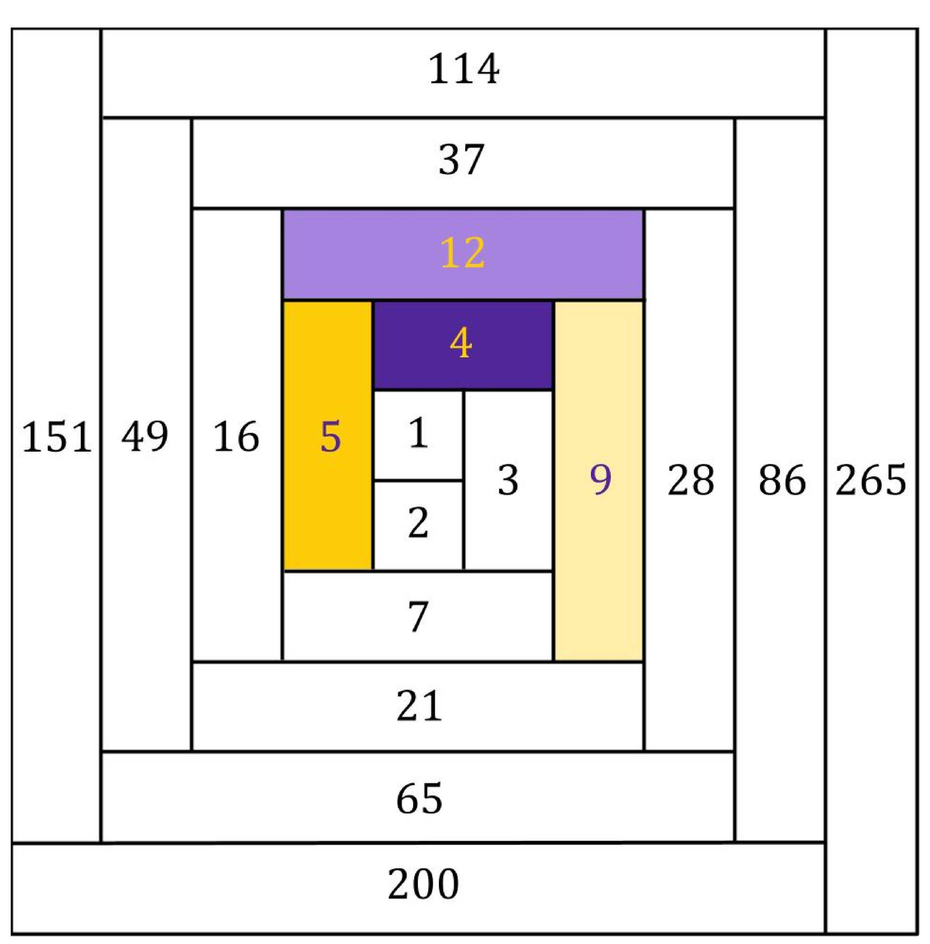

Figure 1: The first 19 terms of the Fibonacci Quilt Sequence, constructed on the Log Cabin quilt pattern. Starting with 1 in the center and for each subsequent term adding the smallest integer that cannot be expressed as the sum of non-adjacent previous terms.

Given an increasing sequence of positive integers $q_{i_{i=1}}^{\infty}$. A decomposition of an integer

$$m = q_{l_1} + q_{l_2} + \cdots + q_{l_t}$$

(where $q_{l_i} > q_{l_{i+1}}$) is an *FQ-legal decomposition*[1] if for all i, j, $|l_i - l_j| \neq 0, 1, 3, 4$ and $\{1, 3\} \not\subset \{l_1, l_2, \ldots, l_t\}$.

Definition[1]

The Fibonacci Quilt Sequence is an increasing sequence of positive integers $\{q_i\}_{i=1}^{\infty}$, where every q_i $(i \geq 1)$ is the smallest positive integer that does not have an FQ-legal decomposition using the elements $\{q_1, \ldots, q_{i-1}\}$.

Theorem[1]

Let q_n denote the n^{th} term in the Fibonacci Quilt, then

$$\text{for } n \geq 5, \quad q_{n+1} = q_{n-1} + q_{n-2},$$

$$\text{for } n \geq 6, \quad q_{n+1} = q_n + q_{n-4}.$$

The Game

The Fibonacci Quilt Game was inspired by the Zeckendorf Game[2]. It is a two player game, beginning with n $1's$, where players alternate applying the following rules until no further moves can be made. The winner is the player who makes the last move.

Rule 1: For $n \geq 2, q_n + q_{n+1} \to q_{n+3}$
Base Rule: $1 + 2 \to 3$

Rule 2: For $n \geq 2, q_n + q_{n+4} \to q_{n+5}$
Base Rule: $1 + 5 \to 4 + 2$ ★

Rule 3: For $n \geq 7$, $2q_n \to q_{n+2} + q_{n-5}$
Base Rules: $1 + 1 \to 2$, $2 + 2 \to 4$,
$3 + 3 \to 2 + 4$, $5 + 5 \to 1 + 9$
$4 + 4 \to 3 + 5$ or $4 + 4 \to 1 + 7$ ★
$7 + 7 \to 5 + 9$ or $7 + 7 \to 2 + 12$ ★

Rule 4: For $n \geq 7$, $q_n + q_{n+3} \to q_{n-5} + q_{n+4}$
Base Rules: $1 + 4 \to 5$, $2 + 5 \to 7$,
$3 + 7 \to 1 + 9$, $4 + 9 \to 1 + 12$
$5 + 12 \to 1 + 16$, $7 + 16 \to 2 + 21$

Rule 5: $1 + 3 \to 4$

★ This rule may only be applied when no other moves are possible.

★ The player may decide which move to make.

On Figure 1 each illegal distance from 4 is colored. Each rule corrects a specific illegal pair of terms in the current decomposition. For example, Rule 1 would take $4 + 5 \to 9$.

Example Game on $n = 8$

We will use the notation that $1^6 \wedge 2^3$ means 6 $1's$ and 3 $2's$. Start with 1^8.

Player 1 applies Rule 3: $1^6 \wedge 2$.
Player 2 applies Rule 1: $1^5 \wedge 3$.
Player 1 applies Rule 5: $1^4 \wedge 4$.
Player 2 applies Rule 4: $1^3 \wedge 5$.
Player 1 applies Rule 3: $1 \wedge 2 \wedge 5$.
Player 2 applies Rule 4: $1 \wedge 7$.
Player 2 wins.

The Game is Playable

Theorem

Every game terminates in a finite number of moves at an FQ-legal decomposition.

To prove this we use a monovarient, the sum of the square roots of the indices of the terms. Each turn decreases this since the following

Rule 1: $\sqrt{n+3} - \sqrt{n} - \sqrt{n+1}$
Rule 2: $\sqrt{n+5} - \sqrt{n} - \sqrt{n+4}$
Rule 3: $\sqrt{n+2} + \sqrt{n-5} - 2\sqrt{n}$
Rule 4: $\sqrt{n+4} + \sqrt{n-5} - \sqrt{n} - \sqrt{n+3}$

are all negative, as well as the base rules, with one exception: $1 + 5 \to 2 + 4$★, thus we restrict the use of this rule.

Lower Bound on Game Length

Notation: Let $L(n)$ denote the maximum number of terms in an FQ-legal decomposition of n. Let $l(n)$ denote the minimum number of terms in an FQ-legal decomposition of n.

Example: $50 = 1 + 49 = 2 + 4 + 16 + 28$
$L(50) = 4, l(50) = 2$.

Theorem

The shortest possible game on n is completed in $n - L(n)$ moves.

Distribution of Game Length

Conjecture

As n goes to infinity, the number of moves in a random game, when all legal moves are equally likely, converges to a Gaussian.

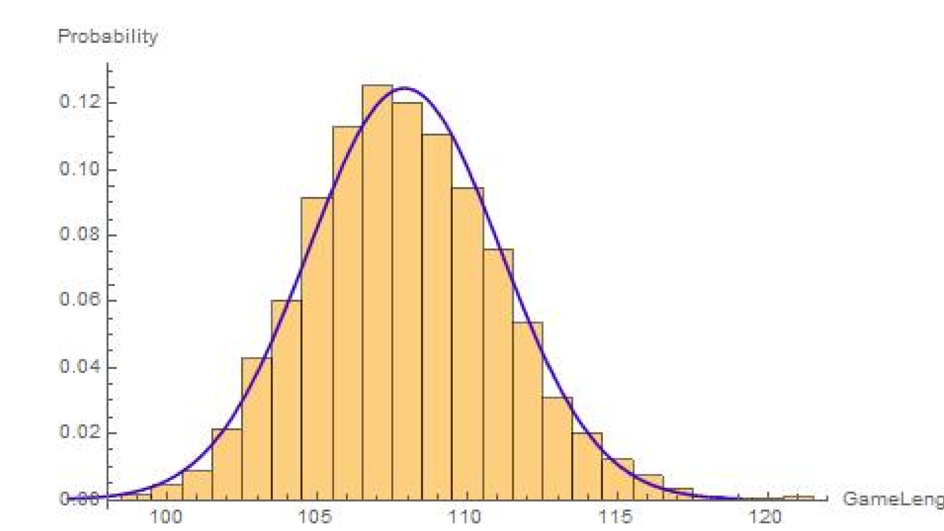

Figure 2: The distribution of 5000 random games on $n = 100$, and the Gaussian curve with the same mean and standard deviation.

Moment	Random Game	Gaussian	Percent Difference
2	11653.9	11653.9	0
4	1.36314×10^8	1.36311×10^8	2.5253×10^{-3}
6	1.60017×10^{12}	1.59997×10^{12}	0.0128609

Figure 3: The moments of our distribution compared to the Gaussian curve with the same mean and standard deviation, calculated in Mathematica.

References

1. M. Catral, P.L. Ford, P.E. Harris, S.J. Miller, D. Nelson, *Legal Decomposition Arising From Non-Positive Linear Recurrences*. Fibonacci Quarterly (54 (2016), no. 4, 348-365).
2. P. Baird-Smith, A. Epstein, K. Flint, S.J. Miller, *The Zeckendorf Game*. (2018).

Acknowledgements

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