The Fibonacci Quilt Game

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Mentored by Steven J. Miller.

Women in Mathematics in New England
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Outline

1. History
2. The Fibonacci Quilt Sequence
3. The Game
4. Game Length
5. Future Work
The Fibonacci Sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55...
The Fibonacci Sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55...

Let $F_0 = F_1 = 1$, and for $n \geq 2$

$$F_n = F_{n-1} + F_{n-2}$$
The Fibonacci Sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55...

Let $F_0 = F_1 = 1$, and for $n \geq 2$

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**Theorem (Zeckendorf)**

Every positive integer can be written uniquely as the sum of non-consecutive Fibonacci numbers where

$$F_n = F_{n-1} + F_{n-2}$$

and $F_1 = 1$, $F_2 = 2$. 
The Fibonacci Quilt Sequence
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The Fibonacci Quilt Sequence
The Fibonacci Quilt Sequence
FQ-legal Decomposition

**Definition (Catral, Ford, Harris, Miller, Nelson)**

Let an increasing sequence of positive integers $q_{i=1}^\infty$ be given. We declare a decomposition of an integer

$$m = q_{l_1} + q_{l_2} + \cdots + q_{l_t}$$

(where $q_{l_i} > q_{l_{i+1}}$) to be an FQ-legal decomposition if for all $i, j$, $|l_i - l_j| \neq 0, 1, 3, 4$ and $\{1, 3\} \not\subset \{l_1, l_2, \ldots, l_t\}$.
The Fibonacci Quilt Sequence

Definition (Catral, Ford, Harris, Miller, Nelson)

The Fibonacci Quilt sequence is an increasing sequence of positive integers \( \{q_i\}_{i=1}^{\infty} \), where every \( q_i \) \((i \geq 1)\) is the smallest positive integer that does not have an FQ-legal decomposition using the elements \( \{q_1, \ldots, q_{i-1}\} \).
Recurrence Relations

Theorem (Catral, Ford, Harris, Miller, Nelson)

Let \( q_n \) denote the \( n^{th} \) term in the Fibonacci Quilt, then

- for \( n \geq 5 \), \( q_{n+1} = q_{n-1} + q_{n-2} \),
- for \( n \geq 6 \), \( q_{n+1} = q_n + q_{n-4} \).
General Rules

- Inspired by the Two Player Zeckendorf Game
- Two players alternate turns, the last person to move wins
- Start the game with $n$ 1’s ($q_1$’s)
- A turn consists of one of 5 rules, which preserve that $\sum q_i = n$ by exchanging a pair $q_i, q_j$ such that $i, j$ are an illegal distance apart for a single term or legal pair.
Rule 1

For $n \geq 2$, $q_n + q_{n+1} \rightarrow q_{n+3}$
General Rules

Rule 2

For $n \geq 2$, $q_n + q_{n+4} \rightarrow q_{n+5}$
General Rules

Rule 3

For $n \geq 7$, $2q_n \rightarrow q_{n+2} + q_{n-5}$
General Rules

**Rule 4**

For $n \geq 7$, $q_n + q_{n+3} \rightarrow q_{n-5} + q_{n+4}$
General Rules

Rule 5

\[ q_1 + q_3 \rightarrow q_4 \]
General Rules

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To handle base cases, we added additional base rules that

- preserves \( \sum q_i = n \)
- does not produce violation of legality
General Rules

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- preserves \( \sum q_i = n \)
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Special Rule

\[ 1 + 5 \rightarrow 2 + 4 \]

Note: This rule can only be applied when nothing else can be done.
Example Game

1, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28...

\[ n = 10 = 9 + 1 \]

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 7 & 9 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
Example Game

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n = 10 = 9 + 1

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Rule 3: \( q_1^2 \rightarrow q_2 \)
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Rule 4: \( q_3 + q_6 \rightarrow q_1 + q_7 \)
The Game is Playable

Theorem

Every game terminates in a finite number of moves at an FQ-legal decomposition.
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\[ q_n \land q_{n+1} \rightarrow q_{n+3}: \sqrt{n + 3} - \sqrt{n} - \sqrt{n + 1} < 0 \]
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q_n \land q_{n+1} \rightarrow q_{n+3}: \sqrt{n + 3} - \sqrt{n} - \sqrt{n + 1} < 0 \\
q_n \land q_{n+4} \rightarrow q_{n+5}: \sqrt{n + 5} - \sqrt{n} - \sqrt{n + 4} < 0
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The Game is Playable

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**Proof Sketch**: The sum of the square roots of the indices of the terms is a strict monovariant.

- 
  \[ q_n \land q_{n+1} \rightarrow q_{n+3} : \sqrt{n + 3} - \sqrt{n} - \sqrt{n + 1} < 0 \]
- 
  \[ q_n \land q_{n+4} \rightarrow q_{n+5} : \sqrt{n + 5} - \sqrt{n} - \sqrt{n + 4} < 0 \]
- 
  \[ 2q_n \rightarrow q_{n+2} \land q_{n-5} : \sqrt{n + 2} + \sqrt{n - 5} - 2\sqrt{n} < 0 \]
The Game is Playable

Theorem

Every game terminates in a finite number of moves at an FQ-legal decomposition.

Proof Sketch: The sum of the square roots of the indices of the terms is a strict monovariant.

- \( q_n \land q_{n+1} \rightarrow q_{n+3}: \sqrt{n+3} - \sqrt{n} - \sqrt{n+1} < 0 \)
- \( q_n \land q_{n+4} \rightarrow q_{n+5}: \sqrt{n+5} - \sqrt{n} - \sqrt{n+4} < 0 \)
- \( 2q_n \rightarrow q_{n+2} \land q_{n-5}: \sqrt{n+2} + \sqrt{n-5} - 2\sqrt{n} < 0 \)
- \( q_n \land q_{n+3} \rightarrow q_{n+4} \land q_{n-5}: \sqrt{n+4} + \sqrt{n-5} - \sqrt{n} - \sqrt{n+3} < 0 \)
Other Results

**Theorem**

There is more than one possible game for any \( n > 3 \).
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There are games of even and odd length for any \( n > 5 \).
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**Theorem**
There are games of even and odd length for any $n > 5$.

**Conjecture**
The game is fair.
Lower Bound on Game Length

Notation
Let $L(n)$ denote the maximum number of terms in an FQ-legal decomposition of $n$. Let $l(n)$ denote the minimum number of terms in an FQ-legal decomposition of $n$.

Examples:
$20 = 16 + 4 = 12 + 7 + 1$
$L(20) = 3, \ l(20) = 2$
## Lower Bound on Game Length

### Notation

Let $L(n)$ denote the maximum number of terms in an FQ-legal decomposition of $n$. Let $l(n)$ denote the minimum number of terms in an FQ-legal decomposition of $n$.

Examples:

- $20 = 16 + 4 = 12 + 7 + 1$
  - $L(20) = 3$, $l(20) = 2$
- $50 = 49 + 1 = 28 + 16 + 4 + 2$
  - $L(50) = 4$, $l(50) = 2$
Lower Bound on Game Length

**Theorem**
The shortest possible game on n is completed in \( n - L(n) \) moves.
Theorem

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Proof Sketch: Strong induction on n.
Lower Bound on Game Length

**Theorem**

The shortest possible game on n is completed in $n - L(n)$ moves.

**Proof Sketch:** Strong induction on n.
If n is in the Fibonacci Quilt Sequence, denoted $q_i$

$$q_{i-3} + q_{i-2} = q_i$$
Lower Bound on Game Length

**Theorem**

The shortest possible game on n is completed in $n - L(n)$ moves.

**Proof Sketch:** Strong induction on n.

If n is in the Fibonacci Quilt Sequence, denoted $q_i$

$$q_{i-3} + q_{i-2} = q_i$$

If n is not in the sequence

$$n = q_{i_1} + q_{i_2} + \cdots + q_{i_{L(n)}}$$

Number of moves:

$$(q_{i_1} - 1) + (q_{i_2} - 1) + \cdots + (q_{i_{L(n)}} - 1)$$

$$= (q_{i_1} + q_{i_2} + \cdots + q_{i_{L(n)}}) - L(n)$$

$$= n - L(n)$$
Conjecture

The distribution of a random game length approaches Gaussian as n increases.

**Figure:** Distribution of 1000 games on n=60
Is there a deterministic game that always results in the lower bound?

What patterns emerge from the winner of certain deterministic games as n increases?

Does either player have a winning strategy?
  - Analogous result on the Zeckendorf Game shows that for $n > 2$, player 2 has a winning strategy
Thank You

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