When Bands Play in Random Matrix Theory

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Background

One of the main interests of **Random Matrix Theory** is the study of the distribution of eigenvalues of matrices under different symmetry constraints. Imposed structures lead to different behavior for the limiting distribution. Certain structured random matrices have been successfully applied to model **zeros of L-functions** and **energy levels of heavy nuclei**.

Motivation

While the eigenvalue density of the family of **real symmetric matrices** converges to a semicircle, different behavior emerges as the symmetry increases. Two different groups completely analyzed the case of **real symmetric Toeplitz matrices** in 2005, seeing a new distribution that is almost Gaussian; this was extended in 2007 to **real symmetric palindromic Toeplitz matrices** (so the first row of the matrix is a palindrome), where the extra symmetry leads to Gaussian behavior.

Why Banded Matrices?

Banded matrices have a long history; band one matrices – not constant on diagonals – are related to the **Laplacian of some systems in mathematical physics**. Furthermore, the behavior of their eigenvalues transitions from that of the density they are drawn from to the semicircle distribution as the number of bands increases from 1 (only the main diagonal) to $D = N$ (the full matrix), with the transition to semicircular behavior essentially complete once $D \approx \sqrt{N}$.

Our Work

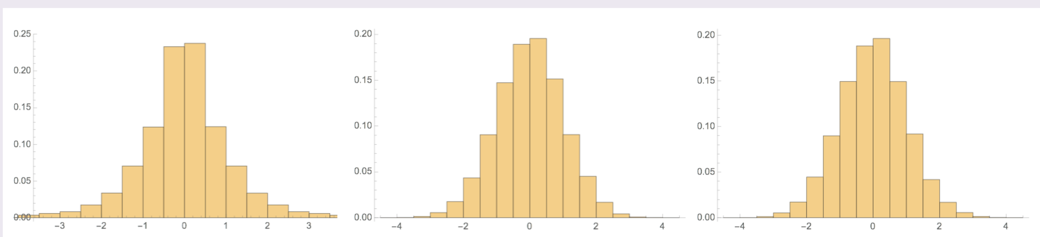
We study a **new case:** $N \times N$ **banded symmetric palindromic Toeplitz matrices (BSPT)** constructed by adding constant diagonals, called bands, to the center and corners of the zero matrix.

If the matrix $A_{N,D}$ has D bands, then it will have non-zero entries on the D diagonals above and below the main diagonal, as well as the corresponding upper and lower corners due to the palindromic constraint.

$$\begin{bmatrix} 0 & a_1 & a_2 & 0 & \dots & 0 & a_2 & a_1 & 0 \\ a_1 & 0 & a_1 & a_2 & 0 & & 0 & a_2 & a_1 \\ a_2 & a_1 & 0 & a_1 & a_2 & \ddots & & 0 & a_2 \\ 0 & a_2 & a_1 & \ddots & \ddots & \ddots & 0 & & 0 \\ \vdots & 0 & a_2 & \ddots & \ddots & \ddots & a_2 & 0 & \vdots \\ 0 & & 0 & \ddots & \ddots & \ddots & a_1 & a_2 & 0 \\ a_2 & 0 & & \ddots & a_2 & a_1 & 0 & a_1 & a_2 \\ a_1 & a_2 & 0 & & 0 & a_2 & a_1 & 0 & a_1 \\ 0 & a_1 & a_2 & 0 & \dots & 0 & a_2 & a_1 & 0 \end{bmatrix}$$

Entries are drawn from probability **distribution p with mean 0, variance 1, and finite higher moments.** $A_{N,D}$ is parameterized by D numbers $(a_1, a_2, \dots, a_D)$.

Below, we show the **graphs** of eigenvalue distribution with **1, 50 and 99 bands** (left to right).



Preliminaries

We define a **probability space** $(\Omega_{N,D}, \mathcal{F}_{N,D}, \mathbb{P}_{N,D})$ of $N \times N$ BSPT matrices with D bands, where

$$\mathbb{P}_{N,D}(\{A_{N,D} : a_i \in [\alpha_i, \beta_i]\}) = \prod_{i=1}^{D} \int_{x_i=\alpha_i}^{\beta_i} p(x_i)\, dx_i$$

To each $A_{N,D}$ we define the **empirical spectral measure** $\mu_{A_{N,D}}$ by affixing a mass of size $\frac{1}{N}$ at each normalized eigenvalue:

$$\mu_{A_{N,D}} = \frac{1}{N} \sum_{i=1}^{N} \delta\left(x - \frac{\lambda_i(A_{N,D})}{\sqrt{2D}}\right)$$

We use the empirical spectral measure to define a **cumulative distribution function**

$$F_{A_{N,D}}(x) = \frac{\{\#i \le N : \frac{\lambda_i(A_{N,D})}{\sqrt{2D}} \le x\}}{N}$$

Our main interest is the behavior of $F^{A_{N,D}/\sqrt{2D}}$ with varying $A_{N,D}$ as $N \to \infty$ and D is fixed.

The k^{th} **moment** of the measure $\mu_{A_{N,D}}$ is

$$M_k(A_{N,D}) = \int_{-\infty}^{\infty} x^k \mu_{A_{N,D}} dx$$
$$= \frac{1}{N} \frac{1}{(2D)^{\frac{k}{2}}} \sum_{i=1}^{N} \lambda_i^k(A_{N,D})$$

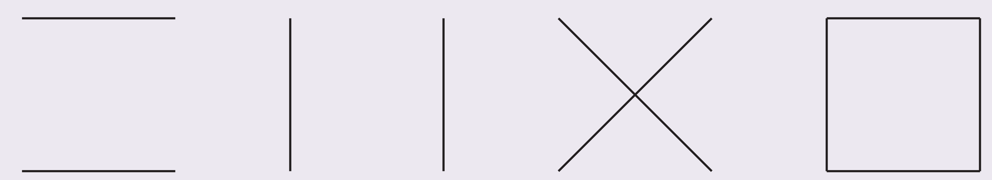
We are interested in the **expectation value** of the moments, denoted $M_{k,D}$.
We cannot compute the moments directly from the eigenvalues since these are unknown. Thus, we use **Eigenvalue Trace Lemma** to compute and bound moments:

$$\sum_{i=1}^{N} \lambda_i^k = \mathrm{Tr}(A_{N,D}^k)$$
$$= \sum_{1 \le i_1, \dots, i_k \le N} a_{i_1, i_2} a_{i_2 i_3} \cdots a_{i_k i_1}.$$

Results

We show that all odd moments vanish and we normalize so that the **second moment** $M_{2,D} = 1$.

Fourth Moment



For a term to contribute, the a's must be matched in either pairs or a quadruple. The total contribution given is

$$M_{4,D} = 3\left(1 - \frac{1}{D}\right)p_2^2 + \left(\frac{1}{2D}\right)p_4$$

where p_k denotes the k'th moment of p.

Sixth Moment

Using the same approach, we derive the sixth moment:

$$15\left(1 - \frac{3}{D} + \frac{2}{D^2}\right)p_2^3 + 15\left(\frac{1}{2D} - \frac{1}{2D^2}\right)p_2 p_4 + \left(\frac{1}{4D^2}\right)p_6$$

Lower Bound

We compute a lower bound for the even moments which can be used to show that the eigenvalue distributions for each D have **unbounded support**:

$$F(D)(2k-1)!! \le M_{2k,D}$$

where $F(D)$ is a positive constant dependent on D.

Upper Bound

We also compute an upper bound for the even moments.

$$M_{2k,D} \le [F_1(D)(2k-1)!! + F_2(D)] \cdot \max(P)$$

where the maximum is taken over all products of moments of p whose orders sum to $2k$. This allows us to apply Carleman's condition to prove that the moments of the eigenvalue distribution determine a **unique distribution**.

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