ABBA and the Random Matrix Discotheque

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Random Matrices?

Definition: Random Matrices

"A random matrix is a matrix ... that is random."

Random Matrix Theory in Quantum Physics

Classical Mechanics: 3-body problem - intractable!

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Schrödinger Equation

$$H\psi_n = E_n\psi_n$$

H: Hamiltonian matrix; entries dependent on system

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Wigner's Insights

- Treat H as random Hermitian matrix

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H: Hamiltonian matrix; entries dependent on system

Wigner's Insights

- Treat H as random Hermitian matrix
- Eigenvalue behavior of H well approximated by averaging over Hermitian ensemble

RMT and Number Theory

Montgomery's Pair Correlation Conjecture:

Spacing between Riemann-zeta function zeros: $1 - \left(\frac{\sin \pi u}{\pi u}\right)^2 + \delta(u)$

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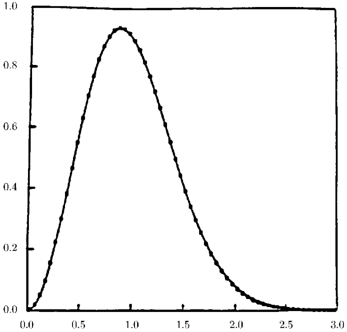
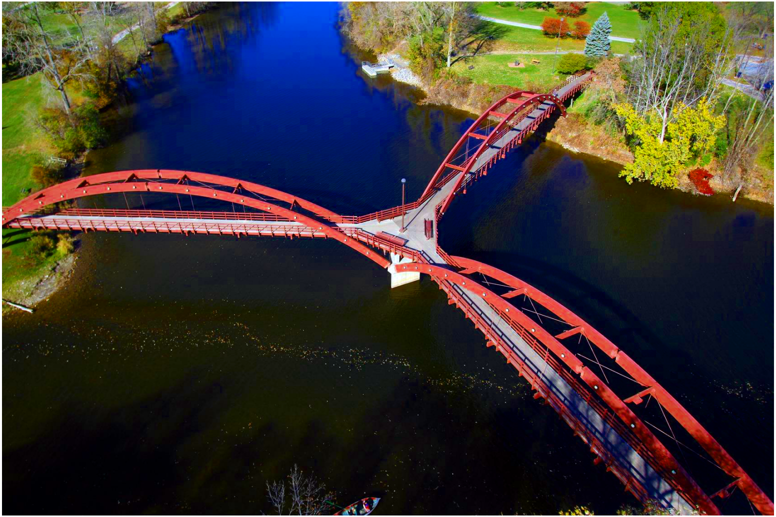


70 million $\zeta(s)$ zero spacings, vs. random matrix prediction (Odlyzko)

Why combine random matrices?



"Tridge" in Midland, Michigan

Why combine random matrices?

	Number Theory	**Random Matrix Theory**
Object	L-functions	Random Matrices
Events	Zeros	Eigenvalues
Process	Rankin-Selberg Convolution	???

Setting the Stage

Component Matrices A, B

Let A, B be $N \times N$ random real symmetric matrices, with entries i.i.d. from a distribution with mean 0, variance 1, and all finite moments.

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Imposed structure on A:

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_2 & a_1 & a_0 \\ a_1 & a_0 & a_1 & \cdots & a_3 & a_2 & a_1 \\ a_2 & a_1 & a_0 & \cdots & a_4 & a_3 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 & \cdots & a_1 & a_0 & a_1 \\ a_0 & a_1 & a_2 & \cdots & a_2 & a_1 & a_0 \end{bmatrix}$$

Symmetric Palindromic Toeplitz (SPT)

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Symmetric Palindromic Toeplitz (SPT)

Disco!

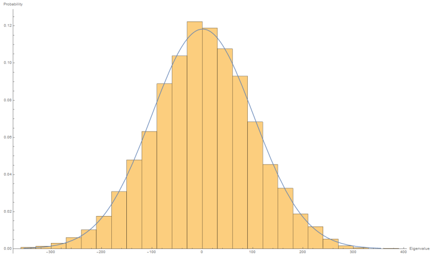
Definition: "Disco" of A, B

$$\mathcal{D}(A,B) = \begin{bmatrix} A & B \\ B & A \end{bmatrix} =$$

Limiting Distributions of A, B

A: Gaussian (Massey, Miller, and Sinsheimer, 2007)

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$



10K × 10K SPT

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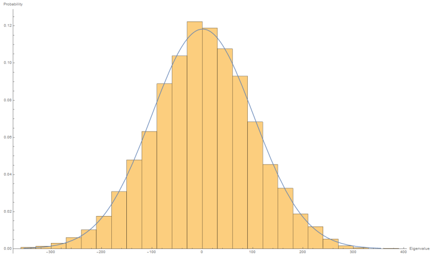
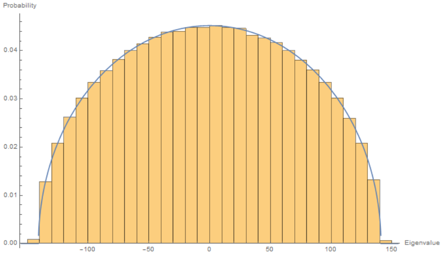


10K × 10K SPT

B: Semi-circle (Wigner, 1955)

$$f(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4 - x^2}, & |x| \le 2 \\ 0, & |x| > 2. \end{cases}$$



10K × 10K Real Symmetric

Defining Probability Space

The kth Moment of D

$$M_k(\mathcal{D}) = \lim_{N\to\infty} \frac{1}{(2N)^{\frac{k}{2}+1}} \mathbb{E}\left[\sum_{i=1}^{2N} \lambda_i^k(\mathcal{D})\right]$$

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Key Equation

$$M_k(\mathcal{D}) = \lim_{N\to\infty} \frac{1}{(2N)^{\frac{k}{2}+1}} \mathbb{E}\left[\text{Trace}(\mathcal{D}^k)\right]$$

Method of Moments

Lemma

$$\begin{aligned}\operatorname{tr}\left(\mathcal{D}^k\right) &= \operatorname{tr}\left((A+B)^k\right) + \operatorname{tr}\left((A-B)^k\right) \\ &= 2\sum_{\substack{l=0 \\ l:\text{even}}}^{k} \sum_{\substack{i_1+\cdots+i_p=k-l \\ j_1+\cdots+j_p=l}} \operatorname{tr}\left(A^{i_1}B^{j_1}\cdots A^{i_p}B^{j_p}\right)\end{aligned}$$

Note: Only the terms with even power survive!

A Simple Example

Take $k = 4$.

$$\begin{aligned} M_4(\mathcal{D}) &= \lim_{N\to\infty} \frac{1}{(2N)^3}\mathbb{E}[\operatorname{tr}\left(\mathcal{D}^4\right)] \\ &= \lim_{N\to\infty} \frac{1}{(2N)^3}\mathbb{E}[\operatorname{tr}\left((A+B)^4\right) + \operatorname{tr}\left((A-B)^4\right)] \\ &= \lim_{N\to\infty} \frac{2}{(2N)^3}\mathbb{E}[\operatorname{tr}\left(A^4\right) + 4\operatorname{tr}\left(A^2B^2\right) + 2\operatorname{tr}(ABAB) + \operatorname{tr}\left(B^4\right)] \end{aligned}$$

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We know

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N^3}\mathbb{E}[\mathrm{tr}\left(A^4\right)] &= M_4(A) \\ \lim_{N\to\infty} \frac{1}{N^3}\mathbb{E}[\mathrm{tr}\left(B^4\right)] &= M_4(B) \end{aligned}$$

Let's Do Some Pairing!

$$\mathrm{tr}\left(A^2B^2\right) = \sum_{1\le i_1,\ldots,i_4\le 4} a_{i_1,i_2}a_{i_2,i_3}b_{i_3,i_4}b_{i_4,i_1}$$

$$\mathrm{tr}(ABAB) = \sum_{1\le i_1,\ldots,i_4\le 4} a_{i_1,i_2}b_{i_2,i_3}a_{i_3,i_4}b_{i_4,i_1}$$

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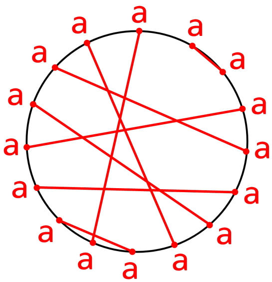
$\mathbb{E}[a] = \mathbb{E}[b] = 0 \implies a, b$ have to be paired

Gaussian



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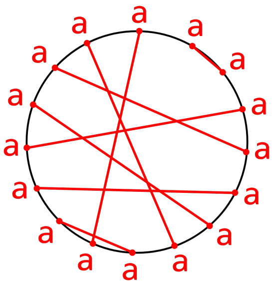
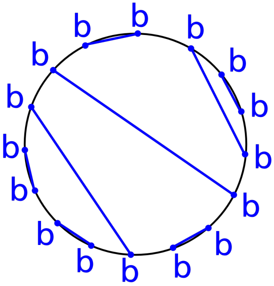


Semicircle



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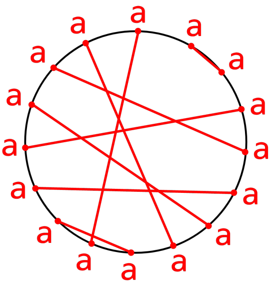

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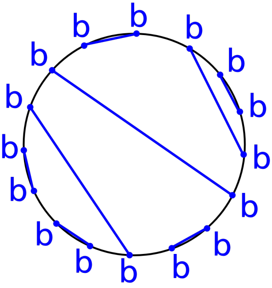
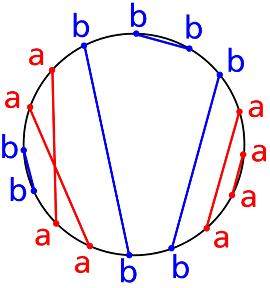


Disco



Let's Do Some Pairing!

Theorem (B., Borade, Devlin, Ma, Miller, and X., 2019)

For 2α a's and 2β b's, the number of contributing pairing configurations is:

$$\mathcal{P}(\alpha,\beta) = \sum_{\substack{|V|=\beta+1 \\ deg(v)=d_1,d_2,\ldots,d_{\beta+1} \\ v\in V}} \frac{2(\alpha+\beta)}{\sigma_r(G)} \prod_{s=1}^{\beta+1} \binom{2r_s+d_s-1}{d_s-1}(2r_s-1)!!$$

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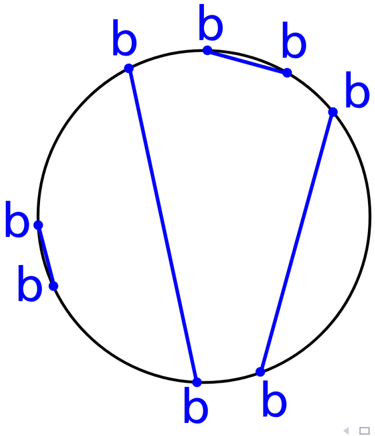

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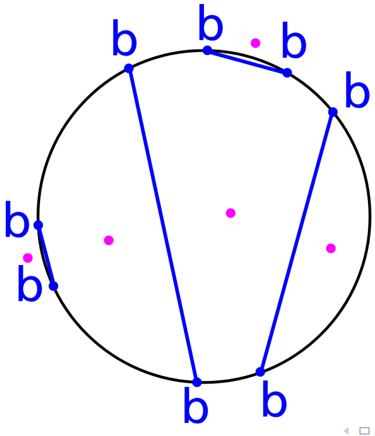

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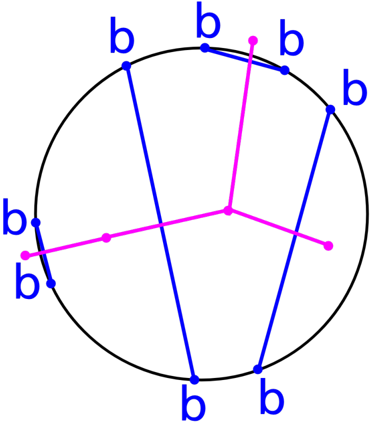

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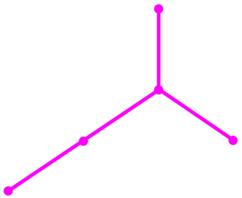

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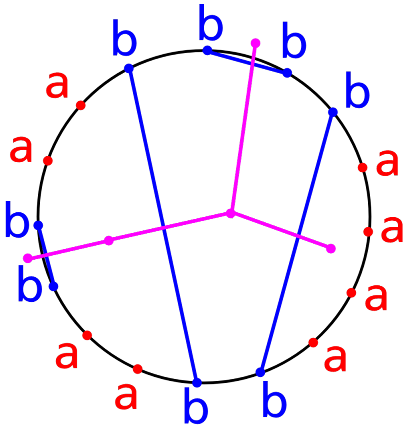

Odd and Small Even Moments

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- Computation of small even moments:

	Semicircle	**Disco of A, B**	**Gaussian**
2	1	1	1
4	2	2.25	3
6	5	7	15
8	14	27.5	105

Upper Bound on Even Moments

Theorem (B., Borade, Devlin, Ma, Miller, and X., 2019)

$$M_{2k}(\mathcal{D}) \leq \frac{(2k-1)!! + \sum_{j=1}^{k-1} \binom{2k}{2j}\binom{2j}{j}\frac{(2k-2j-1)!!}{j+1} + \frac{1}{k+1}\binom{2k}{k}}{2^{k-1}}$$

$$\leq M_{2k}(A)$$

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Sketch:

$$(2k-1)!! \quad \leftrightarrow \quad \text{Contribution of } \mathbb{E}\left[\mathrm{Tr}\left(A^{2k}\right)\right]$$

$$\frac{1}{k+1}\binom{2k}{k} \quad \leftrightarrow \quad \text{Contribution of } \mathbb{E}\left[\mathrm{Tr}\left(B^{2k}\right)\right]$$

$$\binom{2j}{j}\frac{(2k-2j-1)!!}{j+1} \quad \leftrightarrow \quad \text{Contribution of } \mathbb{E}\left[\mathrm{Tr}\left(A^{2k-2j}B^{2j}\right)\right]$$

Lower Bound on Even Moments

Theorem (B., Borade, Devlin, Ma, Miller, and X., 2019)

$$\begin{aligned} M_{2k}(\mathcal{D}) &\geq \frac{(2k-1)!! + \sum_{j=1}^{k-1} 2k\binom{2j}{j}\frac{(2k-2j-1)!!}{j+1} + \frac{1}{k+1}\binom{2k}{k}}{2^{k-1}} \\ &\geq M_{2k}(B) \end{aligned}$$

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Limiting Bounds

Theorem (B., Borade, Devlin, Ma, Miller, and X., 2019)

$$\lim_{k\to\infty} \frac{M_{2k}(\mathcal{D})}{M_{2k}(A)} = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \frac{M_{2k}(B)}{M_{2k}(\mathcal{D})} = 0.$$

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$$\lim_{k\to\infty} \frac{2^{k-1}\frac{1}{k+1}\binom{2k}{k}}{(2k-1)!! + \sum_{j=1}^{k-1} 2k\binom{2j}{j}\frac{(2k-2j-1)!!}{j+1} + \frac{1}{k+1}\binom{2k}{k}} = 0$$

Unbounded Support

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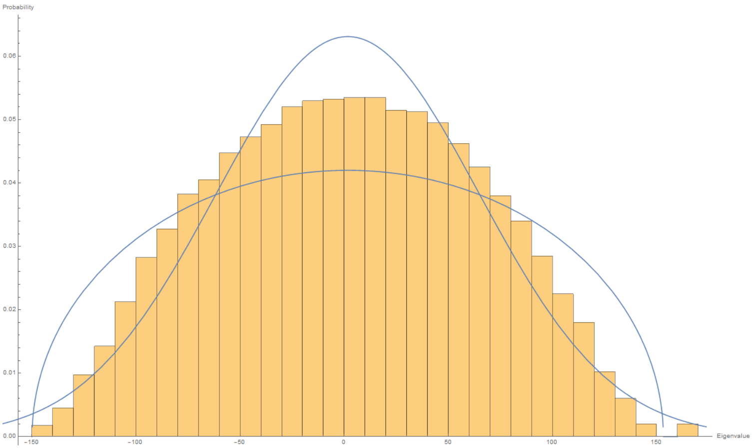
Sketch:
If $\mathrm{supp}(P) \subset [-B, B]$ then $\sqrt[2k]{M_{2k}(\mathcal{D})} \leq B$ for all k.

$$\sqrt[2k]{M_{2k}(\mathcal{D})} \geq \left[\frac{(2k-1)!! + \sum_{j=1}^{k-1} 2k \binom{2j}{j} \frac{(2k-2j-1)!!}{j+1} + \frac{1}{k+1}\binom{2k}{k}}{2^k} \right]^{\frac{1}{2k}}$$

$$> \frac{1}{2}\sqrt{k} \qquad \textit{(Stirling's Formula)}$$

$$\gg B$$

Disco vs. Gaussian, Semicircle



Thank you!