Crescent Configurations Under Non-Euclidean Norms

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Maine-Québec Number Theory Conference, October 2019

Outline

Erdős distinct distances problem

Crescent configurations under Euclidean norms

Crescent configurations under L^p norms

- Line-like configurations in L^p
- Crescent configurations in L^p

Erdős distinct distances problem

Question [Erdős, 1946]

Given n points in a plane, what is the minimum number of distinct distances (n) that they determine?

We \expect" $\binom{n}{2} = O(n^2)$ distinct distances. How low can we go?

Erdős Distinct Distances Problem: Bounds

Upper bounds:

$\Delta(n) = O\left(\frac{n}{\sqrt{\log n}}\right)$ (Erdős, 1946)

Erdős Distinct Distances Problem: Bounds

Upper bounds:

$(\; n) = O(\frac{n}{\log n})$ (Erdős, 1946)

Lower bounds:

$(\; n) = (\; n^{1=2})$ (Erdős, 1946)

$(\; n) = (\; n^{2=3})$ (Moser, 1952)

$(\; n) = (\; n^{5=7})$ (Chung, 1984)

$(\; n) = (\; n^{4=5}= \log n)$ (Chung + Szemerédi + Trotter, 1992)

$(\; n) = (\; n^{4=5})$ (Székely, 1993)

$(\; n) = (\; n^{6=7})$ (Solymosi + Tóth, 2001)

$(\; n) = (\; n^{\frac{4}{5 \;\; 1}}) \quad (\; n^{0:8636})$ (Tardos, 2003)

$(\; n) = (\; n^{\frac{48 \;\; 14}{55 \;\; 16}}) \quad (\; n^{0:8641})$ (Katz + Tardos, 2004)

$(\; n) = (\; \frac{n}{\log n})$ (Guth + Katz, 2015)

Erdős Distinct Distances Problem: Variants

The structure of all near-optimal point sets (which obtain $O(\frac{n}{\sqrt{\log n}})$)

Restriction: no 3 points on a line

Restriction: no 3 points on a line and no 4 points on a circle (general position)

Higher dimensions

Bipartite problems (points lie on one of two lines)

Distinct distances with local properties

Crescent con gurations

Erdős' Question

Question [Erdős, 1989]

Does there exist a set of n points such that:

1. The n points determine $n - 1$ distinct distances
2. For all $1 \leq i \leq n - 1$, there exists a distance which occurs exactly i times

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Answer: Yes!

- n equally spaced points on a line
- n equally spaced points on a circular arc

Erdős' Crescent con gurations

To rule out these trivial con gurations, Erdős introduced an additional requirement that the points lie in general position.

De nition

We say that n points in the plane lie in general position if no three points lie on a common line and no four points lie on a common circle.

Erdős' Crescent configurations

To rule out these trivial configurations, Erdős introduced an additional requirement that the points lie in general position.

Definition

We say that n points in the plane lie in *general position* if no three points lie on a common line and no four points lie on a common circle.

This leads to the definition of a crescent configuration.

Definition

We say that n points in the plane form a *crescent configuration* if:

1. The n points lie in general position
2. The n points determine $n - 1$ distinct distances
3. For all $1 \le i \le n - 1$, there exists a distance which occurs exactly i times

Current results about crescent configurations

For $4 \le n \le 8$, constructions are known (Erdős, I. Palásti, A. Liu, and C. Pomerance).

For $n \ge 9$, it is an open problem whether crescent configurations of size n exist.

Crescent configurations are rare: heuristics

We "expect" crescent configurations to be extremely rare.

Definition

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2. The n points determine $n - 1$ distinct distances
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- By Guth and Katz (2015), n points determine $\Omega(\frac{n}{\log n})$ distinct distances. Just $n - 1$ distinct distances is cutting close!
- The general position condition is very restrictive.
- The multiplicity condition is very restrictive.

L^p norm

We examine how crescent configurations behave under a generalization of the L^2 norm, the L^p norm.

Definition (L^p distance)

Let $1 \le p < \infty$. Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two points in the plane. Their L^p distance is given by:

$$d_p(a, b) = \left(|b_x - a_x|^p + |b_y - a_y|^p\right)^{1/p}$$

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$$d_p(a, b) = (|b_x - a_x|^p + |b_y - a_y|^p)^{1/p}$$

There is also the notion of the L^∞ norm.

Definition (L^∞ distance)

Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two points in the plane. Their L^∞ distance is given by:

$$d_\infty(a, b) = \max\{|b_x - a_x|, |b_y - a_y|\}$$

L^p unit balls and perpendicular bisectors

Unit ball : set of points which have 1 from the origin.

Perpendicular bisector : set of points which are equidistant from two given points.

L^1	L^2	L^3	L^1

Crescent configurations in L^p

Now we can ask the same question about crescent configurations in L^p.

Question [in L^p]

Does there exist a set of n points such that:

1. The n points determine $n-1$ distinct distances
2. For all $1 \leq i \leq n-1$, there exists a distance which occurs exactly i times

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Recall in L^2, we introduced the condition that the points must lie in general position in order to eliminate trivial crescent configurations.

Step 1: For $1 \leq p \leq \infty$, find all trivial crescent configurations in L^p.
Step 2: Introduce a condition in the definition of L^p crescent configurations to eliminate these trivial configurations.

Line-like configurations

Recall the trivial crescent configurations in L^2:

Key observation: The distance graphs of all of these trivial crescent configurations are isomorphic to the distance graph of n equally spaced points on a line.

Line-like configurations

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Key observation: The distance graphs of all of these trivial crescent configurations are isomorphic to the distance graph of n equally spaced points on a line.

Definition

We say that n points in the plane form a *line-like configuration* if their distance graph is isomorphic to the distance graph of n equally spaced points on a line.

L^p crescent con gurations

De nition

We say that n points in the plane form a line-like con guration if their distance graph is isomorphic to the distance graph of n equally spaced points on a line.

The trivial crescent con gurations in L^p are precisely the line-like con gurations.

De nition (L^p crescent con guration)

We say that n points in the plane form a crescent con guration if:

1. The n points do not contain a line-like con guration of size four
2. No three points lie on a line, and no four points lie on a L^p ball
3. The n points determine n 1 distinct distances
4. For all 1 i n 1, there exists a distance which occurs exactly i times

Constructing line-like configurations: A geometrical approach

For $1 \le p \le \infty$, we can construct line-like configurations in L^p using the same general approach.

Constructing line-like con gurations: A geometrical approach

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Constructing line-like configurations: A geometrical approach

For $1 \leq p \leq \infty$, we can construct line-like configurations in L^p using the same general approach.

Constructing line-like configurations: A geometrical approach

For $1 < p < \infty$, we can construct line-like configurations in L^p using the same general approach.

L^p line-like configurations, $p \in (1,\infty) \setminus \{2\}$

Conjecture

For $p \in (1,\infty) \setminus \{2\}$, the only line-like configurations of size $n \geq 5$ are sets of equally spaced points on a line.

Reasoning: We have numerical evidence (Mathematica) which suggest that no other line-like configurations exist. Trying to geometrically construct a line-like configuration which does not lie on a straight line results in near-misses:

L^1 line-like con gurations

We have a large family of L^1 line-like con gurations, for example

We can construct in nitely many L^1 line-like con gurations like this by a geometrical argument:

This construction works for every norm which is not strictly convex.

L^1 line-like configurations

Definition

Line-like crescent configuration

1. No three points lie on a common line.
2. No four points lie on a common L^1 circle.
3. Distance graph is isomorphic to equally spaced points on a line.

L^1 line-like con gurations

straight	screw	stair	twisted
n 3	3 n 4	3 n 6	n 8

L^1 line-like con gurations

Theorem

Let $n \quad 7$. Then every line-like crescent con guration in L^1 of size n is a perpendicular perturbation of a horizontal or vertical line.

L^1 line-like con gurations

Theorem

Let n 7. Then every line-like crescent con guration in L^1 of size n is a perpendicular perturbation of a horizontal or vertical line.

Every line-like con guration of size n 7 in L^1 satis es at least one of the following three properties.

- Three points lie on a common line.
- Four points lie on a common L^1 circle.
- The set of n points is a perpendicular perturbation of a horizontal or vertical line, i.e., has very similar structure to a set of n equally spaced points on a horizontal or vertical line.

Line-like configurations: summary

Our results show that:

- Line-like configurations have four different types of behavior for $p = 1$, $p = 2$, $p \in (1, \infty) \setminus \{2\}$, and $p = \infty$.
- Having an understanding of the line-like configurations in L^p means that we have an understanding of the trivial crescent configurations in L^p.

Crescent configurations in L^p

Definition (L^p crescent configuration)

We say that n points in the plane form a *crescent configuration* if:

1. The n points do not contain a line-like configuration of size four
2. No three points lie on a line, and no four points lie on a L^p ball
3. The n points determine $n - 1$ distinct distances
4. For all $1 \leq i \leq n - 1$, there exists a distance which occurs exactly i times

Recall: Crescent configurations are rare. In L^2, it is an open problem whether crescent configurations of size n exist for $n \geq 9$.

Our Question

In L^p, for which n do there exist crescent configurations of size n?

Crescent con gurations in L^p, $1 < p < 1$

We have a construction for a crescent con guration in L^p of size $n = 4$.

This construction can be generalized to any norm.

Crescent configurations in L^1

We constructed crescent configurations in L^1 of sizes $4, 5, 6, 7$.

Our construction of size 7:

$P_1 = (0; 0)$
$P_2 = (0; 2)$
$P_3 = (2; 2)$
$P_4 = (4; 6)$
$P_5 = (4; 8)$
$P_6 = (5; 1)$
$P_7 = (7; 1)$

Crescent con gurations in L^1

We constructed crescent con gurations in L^1 of sizes $4, 5, 6, 7, 8$.

Our construction of size 8:

$P_1 = (0, 0)$
$P_2 = (0, 6)$
$P_3 = (1, 3)$
$P_4 = (2, 4)$
$P_5 = (3, 2)$
$P_6 = (4, 1)$
$P_7 = (5, 5)$
$P_8 = (6, 7)$

Future Work

Continuations of our work

- Disproving the existence of large (strong) crescent con gurations and large line-like con gurations in most norms
- Constructing crescent con gurations of size 5 in generic norms

Extensions of our work

- Classifying line-like crescent con gurations in non-strictly convex norms
- Generalize the notion of higher dimensional crescent con gurations to arbitrary normed spaces

Acknowledgements

Thanks to

Prof. Steven J. Miller (Mentor, NSF Grant DMS1561945),

Prof. Eyvindur Palsson (Mentor),

the SMALL REU program (NSF grant DMS1659037),

The N.S Reynolds foundation for funding,

and to you, for your attention today!