Crescent Configurations Under Non-Euclidean Norms

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Outline

- Erdős distinct distances problem
- Crescent configurations under Euclidean norms
- Crescent configurations under $L^p$ norms
  - Line-like configurations in $L^p$
  - Crescent configurations in $L^p$
Erdős distinct distances problem

Question [Erdős, 1946]

Given \( n \) points in a plane, what is the minimum number of distinct distances \( \Delta(n) \) that they determine?

We “expect” \( \binom{n}{2} = O(n^2) \) distinct distances. How low can we go?
Erdős Distinct Distances Problem: Bounds

Upper bounds:

- $\Delta(n) = O\left(\frac{n}{\sqrt{\log n}}\right)$ (Erdős, 1946)
Erdős Distinct Distances Problem: Bounds

Upper bounds:
- $\Delta(n) = O\left(\frac{n}{\sqrt{\log n}}\right)$ (Erdős, 1946)

Lower bounds:
- $\Delta(n) = \Omega(n^{1/2})$ (Erdős, 1946)
- $\Delta(n) = \Omega(n^{2/3})$ (Moser, 1952)
- $\Delta(n) = \Omega(n^{5/7})$ (Chung, 1984)
- $\Delta(n) = \Omega\left(n^{4/5}/\log n\right)$ (Chung + Szemerédi + Trotter, 1992)
- $\Delta(n) = \Omega\left(n^{4/5}\right)$ (Székely, 1993)
- $\Delta(n) = \Omega\left(n^{6/7}\right)$ (Solymosi + Tóth, 2001)
- $\Delta(n) = \Omega\left(n^{\frac{4\epsilon}{5\epsilon - 1}}\right) \approx \Omega\left(n^{0.8636}\right)$ (Tardos, 2003)
- $\Delta(n) = \Omega\left(n^{\frac{48 - 14\epsilon}{55 - 16\epsilon}}\right) \approx \Omega\left(n^{0.8641}\right)$ (Katz + Tardos, 2004)
- $\Delta(n) = \Omega\left(\frac{n}{\log n}\right)$ (Guth + Katz, 2015)
Erdős Distinct Distances Problem: Variants

- The structure of all near-optimal point sets (which obtain $O\left(\frac{n}{\sqrt{\log n}}\right)$)
- Restriction: no 3 points on a line
- Restriction: no 3 points on a line and no 4 points on a circle (general position)
- Higher dimensions
- Bipartite problems (points lie on one of two lines)
- Distinct distances with local properties
- Crescent configurations
Erdős’ Question

Question [Erdős, 1989]

Does there exist a set of $n$ points such that:

1. The $n$ points determine $n - 1$ distinct distances
2. For all $1 \leq i \leq n - 1$, there exists a distance which occurs exactly $i$ times
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Answer: Yes!

1. $n$ equally spaced points on a line
2. $n$ equally spaced points on a circular arc
Erdős' Crescent configurations

To rule out these trivial configurations, Erdős introduced an additional requirement that the points lie in general position.

**Definition**

We say that $n$ points in the plane lie in **general position** if no three points lie on a common line and no four points lie on a common circle.
Erdős’ Crescent configurations

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This leads to the definition of a crescent configuration.

**Definition**

We say that \( n \) points in the plane form a **crescent configuration** if:

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2. The \( n \) points determine \( n - 1 \) distinct distances
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Current results about crescent configurations

For $4 \leq n \leq 8$, constructions are known (Erdős, I. Pálásti, A. Liu, and C. Pomerance).

For $n \geq 9$, it is an open problem whether crescent configurations of size $n$ exist.
Crescent configurations are rare: heuristics

We “expect” crescent configurations to be extremely rare.

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- By Guth and Katz (2015), \( n \) points determine \( \Omega\left(\frac{n}{\log n}\right) \) distinct distances. Just \( n - 1 \) distinct distances is cutting close!
- The general position condition is very restrictive.
- The multiplicity condition is very restrictive.
We examine how crescent configurations behave under a generalization of the $L^2$ norm, the $L^p$ norm.

**Definition ($L^p$ distance)**

Let $1 \leq p < \infty$. Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two points in the plane. Their $L^p$ distance is given by:

$$d_p(a, b) = (|b_x - a_x|^p + |b_y - a_y|^p)^{1/p}$$
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There is also the notion of the $L^\infty$ norm.

**Definition ($L^\infty$ distance)**

Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two points in the plane. Their $L^\infty$ distance is given by:

$$d_\infty(a, b) = \max\{|b_x - a_x|, |b_y - a_y|\}$$
**Unit ball**: set of points which have 1 from the origin.

**Perpendicular bisector**: set of points which are equidistant from two given points.

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<th>$L^1$</th>
<th>$L^2$</th>
<th>$L^3$</th>
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$L^1$ unit balls and perpendicular bisectors
Crescent configurations in $L^p$

Now we can ask the same question about crescent configurations in $L^p$.

**Question [in $L^p$]**

Does there exist a set of $n$ points such that:

1. The $n$ points determine $n - 1$ distinct distances
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Recall in $L^2$, we introduced the condition that the points must lie in general position in order to eliminate trivial crescent configurations.

**Step 1**: For $1 \leq p \leq \infty$, find all trivial crescent configurations in $L^p$.

**Step 2**: Introduce a condition in the definition of $L^p$ crescent configurations to eliminate these trivial configurations.
Line-like configurations

Recall the trivial crescent configurations in $L^2$:

![Diagram of line-like configurations]

**Key observation:** The distance graphs of all of these trivial crescent configurations are isomorphic to the distance graph of $n$ equally spaced points on a line.
Line-like configurations

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Key observation: The distance graphs of all of these trivial crescent configurations are isomorphic to the distance graph of $n$ equally spaced points on a line.

Definition
We say that $n$ points in the plane form a line-like configuration if their distance graph is isomorphic to the distance graph of $n$ equally spaced points on a line.
\(L^p\) crescent configurations

**Definition**

We say that \(n\) points in the plane form a **line-like configuration** if their distance graph is isomorphic to the distance graph of \(n\) equally spaced points on a line.

The trivial crescent configurations in \(L^p\) are precisely the line-like configurations.

**Definition (\(L^p\) crescent configuration)**

We say that \(n\) points in the plane form a **crescent configuration** if:

1. The \(n\) points do not contain a line-like configuration of size four
2. No three points lie on a line, and no four points lie on a \(L^p\) ball
3. The \(n\) points determine \(n - 1\) distinct distances
4. For all \(1 \leq i \leq n - 1\), there exists a distance which occurs exactly \(i\) times
For $1 \leq p \leq \infty$, we can construct line-like configurations in $L^p$ using the same general approach.
Constructing line-like configurations: A geometrical approach

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**Conjecture**

For $p \in (1, \infty) \setminus \{2\}$, the only line-like configurations of size $n \geq 5$ are sets of equally spaced points on a line.

Reasoning: We have numerical evidence (Mathematica) which suggests that no other line-like configurations exist. Trying to geometrically construct a line-like configuration which does not lie on a straight line results in near-misses:
We have a large family of $L^1$ line-like configurations, for example

We can construct infinitely many $L^1$ line-like configurations like this by a geometrical argument:

This construction works for every norm which is not strictly convex.
\( L^\infty \) line-like configurations

**Definition**

Line-like crescent configuration

1. No three points lie on a common line.
2. No four points lie on a common \( L^\infty \) circle.
3. Distance graph is isomorphic to \( n \) equally spaced points on a line.
\( L^\infty \) line-like configurations

- Straight: \( n \geq 3 \)
- Screw: \( 3 \leq n \leq 4 \)
- Stair: \( 3 \leq n \leq 6 \)
- Twisted: \( n \leq 8 \)
$L^\infty$ line-like configurations

**Theorem**

Let $n \geq 7$. Then every line-like crescent configuration in $L^\infty$ of size $n$ is a perpendicular perturbation of a horizontal or vertical line.
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Let $n \geq 7$. Then every line-like crescent configuration in $L^\infty$ of size $n$ is a perpendicular perturbation of a horizontal or vertical line.

Every line-like configuration of size $n \geq 7$ in $L^\infty$ satisfies at least one of the following three properties.

1. Three points lie on a common line.
2. Four points lie on a common $L^\infty$ circle.
3. The set of $n$ points is a perpendicular perturbation of a horizontal or vertical line, i.e., has very similar structure to a set of $n$ equally spaced points on a horizontal or vertical line.
Line-like configurations: summary

Our results show that:

1. Line-like configurations have four different types of behavior for $p = 1$, $p = 2$, $p \in (1, \infty) \setminus \{2\}$, and $p = \infty$.

2. Having an understanding of the line-like configurations in $L^p$ means that we have an understanding of the trivial crescent configurations in $L^p$. 
Crescent configurations in $L^p$

**Definition ($L^p$ crescent configuration)**

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2. No three points lie on a line, and no four points lie on a $L^p$ ball
3. The $n$ points determine $n - 1$ distinct distances
4. For all $1 \leq i \leq n - 1$, there exists a distance which occurs exactly $i$ times

Recall: Crescent configurations are rare. In $L^2$, it is an open problem whether crescent configurations of size $n$ exist for $n \geq 9$.

**Our Question**

In $L^p$, for which $n$ do there exist crescent configurations of size $n$?
Crescent configurations in $L^p$, $1 < p < \infty$

We have a construction for a crescent configuration in $L^p$ of size $n = 4$.

This construction can be generalized to any norm.
Crescent configurations in $L^1$

We constructed crescent configurations in $L^1$ of sizes 4, 5, 6, 7.

Our construction of size 7:

$$P_1 = (0, 0)$$
$$P_2 = (0, 2)$$
$$P_3 = (2, 2)$$
$$P_4 = (4, 6)$$
$$P_5 = (4, 8)$$
$$P_6 = (5, 1)$$
$$P_7 = (7, 1)$$
Crescent configurations in $L^\infty$

We constructed crescent configurations in $L^\infty$ of sizes 4, 5, 6, 7, 8.

Our construction of size 8:

$$P_1 = (0, 0)$$
$$P_2 = (0, 6)$$
$$P_3 = (1, 3)$$
$$P_4 = (2, 4)$$
$$P_5 = (3, 2)$$
$$P_6 = (4, 1)$$
$$P_7 = (5, 5)$$
$$P_8 = (6, 7)$$
Future Work

Continuations of our work

- Disproving the existence of large (strong) crescent configurations and large line-like configurations in most norms
- Constructing crescent configurations of size $\geq 5$ in generic norms

Extensions of our work

- Classifying line-like crescent configurations in non-strictly convex norms
- Generalize the notion of higher dimensional crescent configurations to arbitrary normed spaces
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Questions

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