Crescent Configurations Under Non-Euclidean Norms

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Outline

- Erdős distinct distances problem
- Crescent configurations under Euclidean norms
- Crescent configurations under L^p norms
  - Line-like configurations in L^p
  - Crescent configurations in L^p

Erdős distinct distances problem

Question [Erdős, 1946]

Given n points in a plane, what is the minimum number of distinct distances $\Delta(n)$ that they determine?

We "expect" $\binom{n}{2} = O(n^2)$ distinct distances. How low can we go?

Erdős Distinct Distances Problem: Bounds

Upper bounds:

- $\Delta(n) = O(\frac{n}{\sqrt{\log n}})$ (Erdős, 1946)

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Lower bounds:

- $\Delta(n) = \Omega(n^{1/2})$ (Erdős, 1946)
- $\Delta(n) = \Omega(n^{2/3})$ (Moser, 1952)
- $\Delta(n) = \Omega(n^{5/7})$ (Chung, 1984)
- $\Delta(n) = \Omega(n^{4/5}/\log n)$ (Chung + Szemerédi + Trotter, 1992)
- $\Delta(n) = \Omega(n^{4/5})$ (Székely, 1993)
- $\Delta(n) = \Omega(n^{6/7})$ (Solymosi + Tóth, 2001)
- $\Delta(n) = \Omega(n^{\frac{4\epsilon}{5\epsilon-1}}) \approx \Omega(n^{0.8636})$ (Tardos, 2003)
- $\Delta(n) = \Omega(n^{\frac{48-14\epsilon}{55-16\epsilon}}) \approx \Omega(n^{0.8641})$ (Katz + Tardos, 2004)
- $\Delta(n) = \Omega(\frac{n}{\log n})$ (Guth + Katz, 2015)

Erdős Distinct Distances Problem: Variants

- The structure of all near-optimal point sets (which obtain $O(\frac{n}{\sqrt{\log n}})$)
- Restriction: no 3 points on a line
- Restriction: no 3 points on a line and no 4 points on a circle (general position)
- Higher dimensions
- Bipartite problems (points lie on one of two lines)
- Distinct distances with local properties
- Crescent configurations

Erdős' Question

Question [Erdős, 1989]

Does there exist a set of n points such that:

1. The n points determine $n-1$ distinct distances
2. For all $1 \leq i \leq n-1$, there exists a distance which occurs exactly i times

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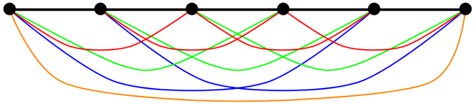
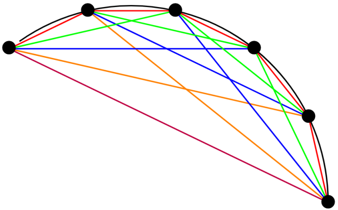
Answer: **Yes!**

1. n equally spaced points on a line
2. n equally spaced points on a circular arc





Erdős' Crescent configurations

To rule out these trivial configurations, Erdős introduced an additional requirement that the points lie in general position.

Definition

We say that n points in the plane lie in **general position** if no three points lie on a common line and no four points lie on a common circle.

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Definition

We say that n points in the plane lie in **general position** if no three points lie on a common line and no four points lie on a common circle.

This leads to the definition of a crescent configuration.

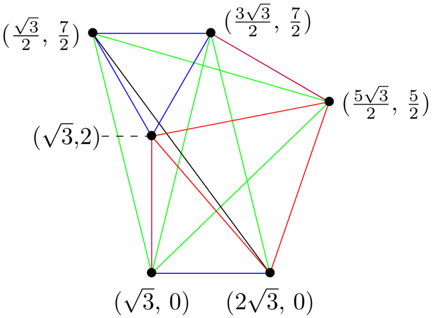
Definition

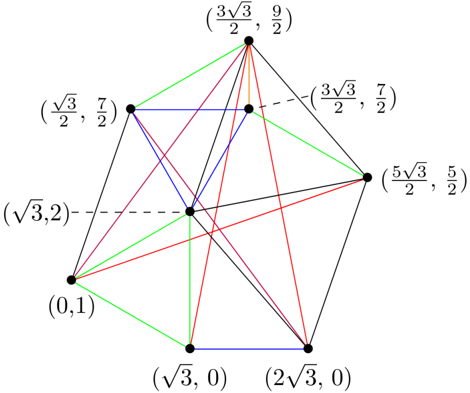
We say that n points in the plane form a **crescent configuration** if:

1. The n points lie in general position
2. The n points determine $n-1$ distinct distances
3. For all $1 \leq i \leq n-1$, there exists a distance which occurs exactly i times

Current results about crescent configurations

For $4 \leq n \leq 8$, constructions are known (Erdős, I. Pàlàsti, A. Liu, and C. Pomerance).





For $n \geq 9$, it is an open problem whether crescent configurations of size n exist.

Crescent configurations are rare: heuristics

We “expect” crescent configurations to be extremely rare.

Definition

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- By Guth and Katz (2015), n points determine $\Omega(\frac{n}{\log n})$ distinct distances. Just $n-1$ distinct distances is cutting close!
- The general position condition is very restrictive.
- The multiplicity condition is very restrictive.

L^p norm

We examine how crescent configurations behave under a generalization of the L^2 norm, the L^p norm.

Definition (L^p distance)

Let $1 \leq p < \infty$. Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two points in the plane. Their L^p **distance** is given by:

$$d_p(a, b) = \left(|b_x - a_x|^p + |b_y - a_y|^p\right)^{1/p}$$

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There is also the notion of the L^∞ norm.

Definition (L^∞ distance)

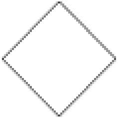
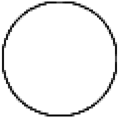
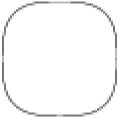
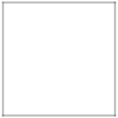
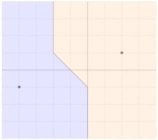
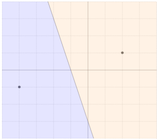
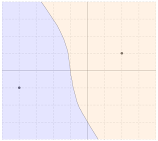
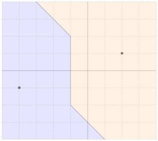
Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two points in the plane. Their L^∞ **distance** is given by:

$$d_\infty(a, b) = \max\{|b_x - a_x|, |b_y - a_y|\}$$

L^p unit balls and perpendicular bisectors

Unit ball: set of points which have 1 from the origin.

Perpendicular bisector: set of points which are equidistant from two given points.

L^1	L^2	L^3	L^∞
			
			

Crescent configurations in L^p

Now we can ask the same question about crescent configurations in L^p.

Question [in L^p]

Does there exist a set of n points such that:

1. The n points determine $n-1$ distinct distances
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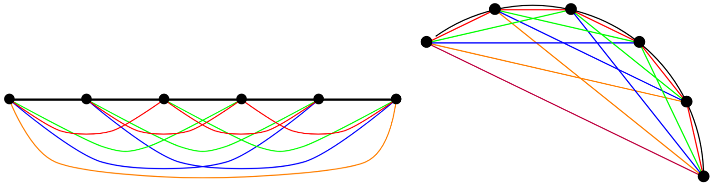
Recall in L^2, we introduced the condition that the points must lie in general position in order to eliminate trivial crescent configurations.

Step 1: For $1 \leq p \leq \infty$, find all trivial crescent configurations in L^p.
Step 2: Introduce a condition in the definition of L^p crescent configurations to eliminate these trivial configurations.

Line-like configurations

Recall the trivial crescent configurations in L^2:



Key observation: The distance graphs of all of these trivial crescent configurations are isomorphic to the distance graph of n equally spaced points on a line.

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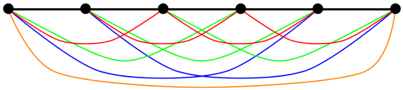

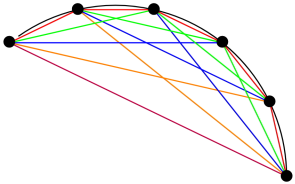

Key observation: The distance graphs of all of these trivial crescent configurations are isomorphic to the distance graph of n equally spaced points on a line.

Definition

We say that n points in the plane form a **line-like configuration** if their distance graph is isomorphic to the distance graph of n equally spaced points on a line.

L^p crescent configurations

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We say that n points in the plane form a **line-like configuration** if their distance graph is isomorphic to the distance graph of n equally spaced points on a line.

The trivial crescent configurations in L^p are precisely the line-like configurations.

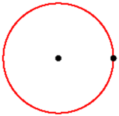
Definition (L^p crescent configuration)

We say that n points in the plane form a **crescent configuration** if:

1. The n points do not contain a line-like configuration of size four
2. No three points lie on a line, and no four points lie on a L^p ball
3. The n points determine $n - 1$ distinct distances
4. For all $1 \leq i \leq n - 1$, there exists a distance which occurs exactly i times

Constructing line-like configurations: A geometrical approach

For $1 \leq p \leq \infty$, we can construct line-like configurations in L^p using the same general approach.



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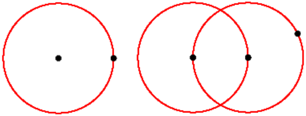

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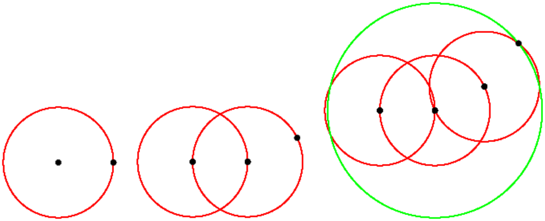

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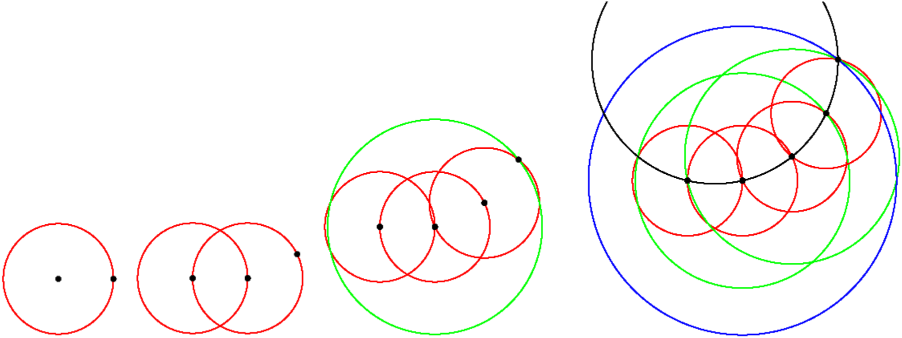
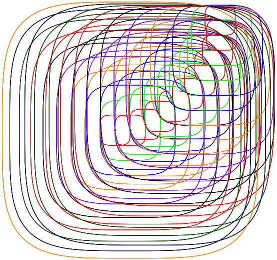


L^p line-like configurations, $p \in (1, \infty) \setminus \{2\}$

Conjecture

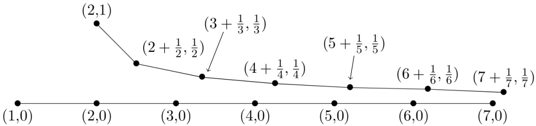
For $p \in (1, \infty) \setminus \{2\}$, the only line-like configurations of size $n \geq 5$ are sets of equally spaced points on a line.

Reasoning: We have numerical evidence (Mathematica) which suggests that no other line-like configurations exist. Trying to geometrically construct a line-like configuration which does not lie on a straight line results in near-misses:

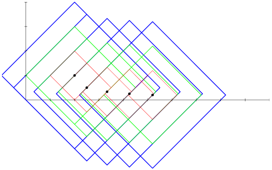


L^1 line-like configurations

We have a large family of L^1 line-like configurations, for example



We can construct infinitely many L^1 line-like configurations like this by a geometrical argument:



This construction works for every norm which is not strictly convex.

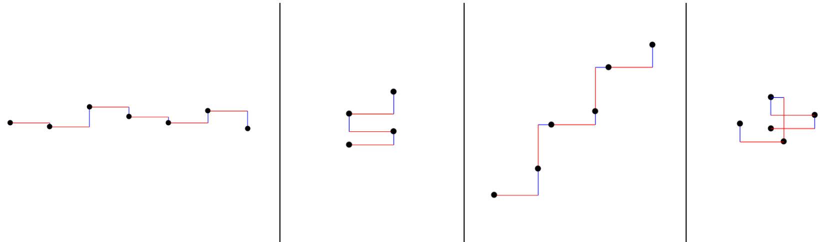
L^∞ line-like configurations

Definition

Line-like crescent configuration

1. No three points lie on a common line.
2. No four points lie on a common L^∞ circle.
3. Distance graph is isomorphic to n equally spaced points on a line.

L^∞ line-like configurations



straight	screw	stair	twisted
$n \geq 3$	$3 \leq n \leq 4$	$3 \leq n \leq 6$	$n \leq 8$

L^∞ line-like configurations

Theorem

Let $n \geq 7$. Then every line-like crescent configuration in L^∞ of size n is a perpendicular perturbation of a horizontal or vertical line.

L^∞ line-like configurations

Theorem

Let $n \geq 7$. Then every line-like crescent configuration in L^∞ of size n is a perpendicular perturbation of a horizontal or vertical line.

Every line-like configuration of size $n \geq 7$ in L^∞ satisfies at least one of the following three properties.

1. Three points lie on a common line.
2. Four points lie on a common L^∞ circle.
3. The set of n points is a perpendicular perturbation of a horizontal or vertical line, i.e., has very similar structure to a set of n equally spaced points on a horizontal or vertical line.

Line-like configurations: summary

Our results show that:

1. Line-like configurations have four different types of behavior for $p = 1$, $p = 2$, $p \in (1, \infty) \setminus \{2\}$, and $p = \infty$.
2. Having an understanding of the line-like configurations in L^p means that we have an understanding of the trivial crescent configurations in L^p.

Crescent configurations in L^p

Definition (L^p crescent configuration)

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1. The n points do not contain a line-like configuration of size four
2. No three points lie on a line, and no four points lie on a L^p ball
3. The n points determine $n-1$ distinct distances
4. For all $1 \leq i \leq n-1$, there exists a distance which occurs exactly i times

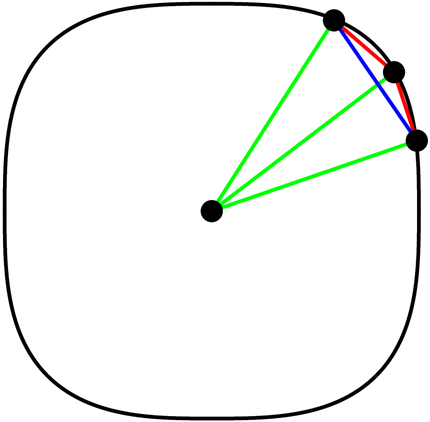
Recall: Crescent configurations are rare. In L^2, it is an open problem whether crescent configurations of size n exist for $n \geq 9$.

Our Question

In L^p, for which n do there exist crescent configurations of size n?

Crescent configurations in L^p, $1 < p < \infty$

We have a construction for a crescent configuration in L^p of size $n = 4$.

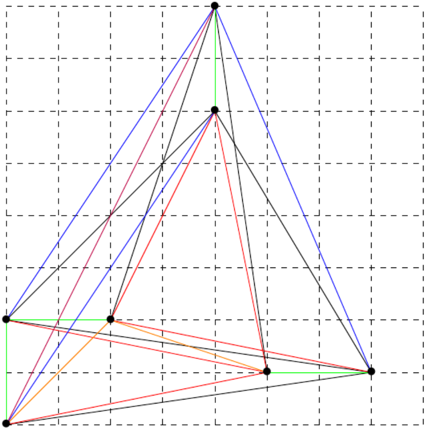


This construction can be generalized to any norm.

Crescent configurations in L^1

We constructed crescent configurations in L^1 of sizes $4, 5, 6, 7$.

Our construction of size 7:

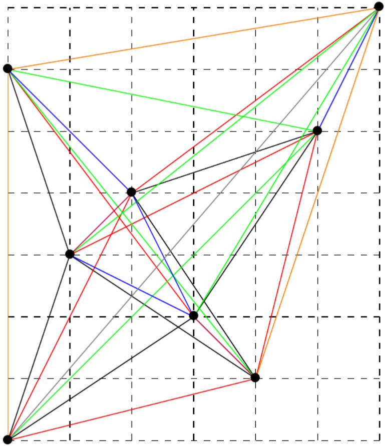


$$P_1 = (0,0)$$
$$P_2 = (0,2)$$
$$P_3 = (2,2)$$
$$P_4 = (4,6)$$
$$P_5 = (4,8)$$
$$P_6 = (5,1)$$
$$P_7 = (7,1)$$

Crescent configurations in L^∞

We constructed crescent configurations in L^∞ of sizes $4, 5, 6, 7, 8$.

Our construction of size 8:



$$P_1 = (0,0)$$
$$P_2 = (0,6)$$
$$P_3 = (1,3)$$
$$P_4 = (2,4)$$
$$P_5 = (3,2)$$
$$P_6 = (4,1)$$
$$P_7 = (5,5)$$
$$P_8 = (6,7)$$

Future Work

Continuations of our work

- Disproving the existence of large (strong) crescent configurations and large line-like configurations in most norms
- Constructing crescent configurations of size ≥ 5 in generic norms

Extensions of our work

- Classifying line-like crescent configurations in non-strictly convex norms
- Generalize the notion of higher dimensional crescent configurations to arbitrary normed spaces

Acknowledgements

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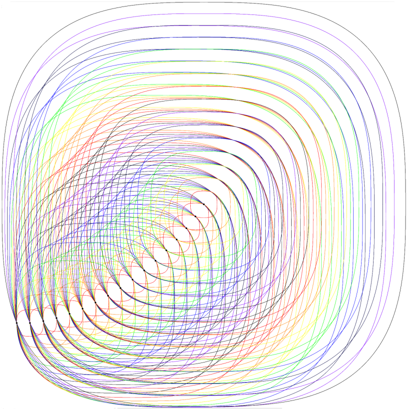
- the SMALL REU program (NSF grant DMS1659037),
- The N.S Reynolds foundation for funding,
- and to you, for your attention today!

Questions



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