Zeros of L-functions near the Central Point and Optimal Test Functions

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  - ◇ nuclear physics, Birch and Swinnerton Dyer conjecture, elliptic curve cryptography.
  - ◇ *n*-level densities, Katz-Sarnak determinants

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  - ◇ n-level densities, Katz-Sarnak determinants

- Bounding the average rank:
  - ◇ Saturated bound for 1-dimensional case.
  - ◇ Extending to give a strong bound for $n \geq 2$ dimensions.
  - ◇ A more general approach to extend and obtain a tighter bound.

Example: Riemann Zeta Function

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \text{ for } \Re(s) > 1.$$

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Riemann Hypothesis

All nontrivial zeros (not negative even integers) of ζ are of the form $\gamma = \frac{1}{2} + i\sigma$ with $\sigma \in \mathbb{R}$.

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- meromorphic continuation to $\mathbb{C}$, of finite order, at most finitely may poles (all on the line $\Re(s) = 1$).
- Functional equation: $\omega \in \mathbb{R}$, $G(s)$ product of Γ-fns:

$$e^{i\omega} G(s) L(s,f) = e^{-i\omega} \overline{G(1-\overline{s}) L(1-\overline{s})}.$$

Applications

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- zeros $\longleftrightarrow$ primes.
- Birch and Swinnerton-Dyer conjecture.
  $\diamond$ Elliptic curve cryptography.

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- Ensembles of matrices (e.g. real symmetric, Hermitian) with entries drawn from probability distribution.
- Study distribution of normalized eigenvalues for given ensemble.

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Applications

- Energy levels of heavy nuclei.
- Elliptic curve cryptography.

1-level Density

Family of Dirichlet L-functions $L(s, f)$ indexed by cuspidal newform $f \in \mathcal{F}$. Riemann hypothesis $\Longrightarrow$ zeros of $L(s, f)$ are of the form $\rho_f = \frac{1}{2} + \mathrm{i}\gamma_f$ with $\gamma_f \in \mathbb{R}$.

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1-level Density

$D(f; \phi) := \sum_{\gamma_f} \phi(\frac{\gamma_f}{2\pi} \log(c_f))$ where $\phi \geq 0$ is even, Schwartz, $\hat{\phi}$ compactly supported, $\phi(0) > 0$. $c_f > 1$ is the analytic conductor.

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Idea:

Varying ϕ, $D(f; \phi)$ measures density of zeros of $L(s, f)$ near central point $s = \frac{1}{2}$.

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Then take a limit:

$$\lim_{Q\to\infty} \mathbb{E}(\mathcal{F}(Q);\phi) = \int_{-\infty}^{\infty} \phi(x) W(\mathcal{F})(x)\,\mathrm{d}x$$

where $W(\mathcal{F})$ is a distribution depending on $\mathcal{F}$.

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From RMT, $W(\mathcal{F})$ is dependent on a symmetry group $G = G(\mathcal{F})$ of $\mathcal{F}$. Write $W(\mathcal{F}) = W_{1,G}$. Some examples:

$$W_{1,O}(x) = 1 + \frac{1}{2}\delta(x)$$

$$W_{1,\mathrm{SO(E)}}(x) = 1 + \frac{\sin(2\pi x)}{2\pi x}$$

$$W_{1,\mathrm{SO(O)}}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x} + \delta(x)$$

1-level Density

Quantity of interest

$\lim_{Q\to\infty}$ $\mathrm{AveRank}(\mathcal{F}(Q))$, where $\mathrm{AveRank}(\mathcal{F}(Q))$ is average order of vanishing of the L-functions with $f \in \mathcal{F}(Q)$.

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Can show that

$$\lim_{Q\to\infty} \text{AveRank}(\mathcal{F}(Q)) \le \frac{\int_{-\infty}^{\infty} \phi(x) W_{1,G}(x)\, \mathrm{d}x}{\phi(0)}$$

n-level Density

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$$D_n(f;\phi) := \sum_{\substack{\gamma_{j,f} \\ |j| \text{ distinct}}} \phi\left(\frac{\gamma_{1,f}}{2\pi}\log(c_f), \frac{\gamma_{2,f}}{2\pi}\log(c_f), \dots, \frac{\gamma_{n,f}}{2\pi}\log(c_f)\right)$$

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Higher Dimensional Bound

$$\lim_{Q\to\infty} \text{WeightedAveRank}(\mathcal{F}(Q)) \leq \frac{\int_{\mathbb{R}^n} \phi(x) W_{n,G}(x)\, dx_1 \cdots dx_n}{\phi(\mathbf{0})}$$

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Goal

Higher level densities give stronger bound. Minimize right-hand side over admissible ϕ for n as large as possible.

Katz-Sarnak Determinants

Set $K_\epsilon(x,y) := \frac{\sin(\pi(x-y))}{\pi(x-y)} + \epsilon\frac{\sin(\pi(x+y))}{\pi(x+y)}$, $\epsilon \in \{0, \pm 1\}$. n-level weights for classical compact groups are:

$$W_{n,\mathrm{SO(E)}}(x) = \det\left(K_1(x_i,x_j)\right)_{i,j\le n}$$

$$W_{n,\mathrm{SO(O)}}(x) = \det\left(K_{-1}(x_i,x_j)\right)_{i,j\le n} + \sum_{k=1}^{n}\delta(x_k)\det\left(K_{-1}(x_i,x_j)\right)_{i,j\ne k}$$

$$W_{n,O}(x) = \frac{1}{2}W_{n,\mathrm{SO(E)}}(x) + \frac{1}{2}W_{n,\mathrm{SO(O)}}(x)$$

$$W_{n,\mathrm{U}}(x) = \det\left(K_0(x_i,x_j)\right)_{i,j\le n}$$

$$W_{n,\mathrm{Sp}}(x) = \det\left(K_{-1}(x_i,x_j)\right)_{i,j\le n}$$

Main Results

Main Idea

Restrict domain to only those ϕ which are linear combinations of single variable test functions:
$$\phi(x) = \sum_{j=1}^{m} \phi_{j,1}(x_1) \cdots \phi_{j,n}(x_n).$$

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Main Result 1

1. Choosing first $n-1$ terms $\phi_{j,1}, \ldots, \phi_{j,n-1}$ carefully, can integrate first $n-1$ variables to obtain new weight function of same form as 1-dimensional weights.
2. 1-level case already solved, so choose ϕ_n optimally for new weight.

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Restrict domain to only those ϕ which are linear combinations of single variable test functions:

$$\phi(x) = \sum_{j=1}^{m} \phi_{j,1}(x_1) \cdots \phi_{j,n}(x_n).$$

Main Result 2

1. Using functional analysis, reduce to a problem similar to the 1-level case.
2. Deal with the added difficulties of poorly behaved Fourier transforms of weight functions.
3. Obtain stronger bound than Main Result 1.

1-level Case

2 Steps

1. Reduce problem to different functional analysis optimization problem.

1-level Case

2 Steps

1. Reduce problem to different functional analysis optimization problem.
2. Use functional analysis tools to solve reduced problem.

Step 1: Reduce Problem

Assume $\mathrm{supp}(\hat{\phi}) \subset [-1, 1]$. Plancherel on numerator, taking then inverting Fourier transform in denominator:

$$\frac{\int_{-\infty}^{\infty} \phi(x) W_{1,G}(x)\, \mathrm{d}x}{\phi(\mathbf{0})} = \frac{\int_{-1}^{1} \hat{\phi}(\xi) \widehat{W_{1,G}}(\xi)\, \mathrm{d}\xi}{\int_{-1}^{1} \hat{\phi}(\xi)\, \mathrm{d}\xi}.$$

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Ahiezer's Theorem and the Paley-Wiener Theorem show ϕ admissible $\iff$ $\hat{\phi}(\xi) = (g * \breve{g})(\xi)$ for some $g \in L^2[-\frac{1}{2}, \frac{1}{2}]$, where $\breve{g}(\xi) = \overline{g(-\xi)}$.

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Some calculations show for $G =$ classical compact group, $\widehat{W_{1,G}}(\xi) = \delta(\xi) + m(\xi)$ on $[-1, 1]$, with $m(\xi)$ real, piecewise continuous, even.

Step 1: Reduce Problem

Some functional analysis: define compact, self-adjoint linear operator $K : L^2[-\frac{1}{2}, \frac{1}{2}] \to L^2[-\frac{1}{2}, \frac{1}{2}]$

$$(Kg)(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} m(x-y)g(y)\,\mathrm{d}y.$$

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Some manipulations:

$$\frac{\int_{-1}^{1} \hat{\phi}(\xi)\widehat{W_{1,G}}(\xi)\,\mathrm{d}\xi}{\int_{-1}^{1} \hat{\phi}(\xi)\,\mathrm{d}\xi} = \frac{\int_{-1}^{1}(g*\breve{g})(\xi)(\delta(\xi)+m(\xi))\,\mathrm{d}\xi}{\int_{-1}^{1}(g*\breve{g})(\xi)\,\mathrm{d}\xi}$$

Step 1: Reduce Problem

$$= \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{1} \left(\delta(\xi)g(\xi+y)\overline{g(y)} + m(\xi)g(\xi+y)\overline{g(y)}\right) \,\mathrm{d}\xi\,\mathrm{d}y}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{1} g(\xi+y)\overline{g(y)}\,\mathrm{d}\xi\,\mathrm{d}y}$$

$$= \frac{\langle g, g\rangle_{L^2} + \int_{-1}^{1} \int_{-\frac{1}{2}+\xi}^{\frac{1}{2}+\xi} m(\xi)g(y)\overline{g(-\xi+y)}\,\mathrm{d}y\,\mathrm{d}\xi}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{1} g(\xi+y)\,\mathrm{d}\xi\,\overline{g(y)}\,\mathrm{d}y}$$

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$$= \frac{\langle g, g\rangle_{L^2} + \langle Kg, g\rangle_{L^2}}{\langle g, 1\rangle_{L^2}\langle 1, g\rangle_{L^2}}$$

$$= \frac{\langle (I + K)g, g\rangle_{L^2}}{|\langle 1, g\rangle_{L^2}|^2}$$

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New Problem

Defining $R : L^2[-\frac{1}{2}, \frac{1}{2}] \to L^2[-\frac{1}{2}, \frac{1}{2}]$ by $R(g) := \frac{\langle (I+K)g, g\rangle_{L^2}}{|\langle 1, g\rangle_{L^2}|^2}$, minimize R over subset of $L^2[-\frac{1}{2}, \frac{1}{2}]$ with denominator $\neq 0$.

Step 2: Minimization

Some observations:

- $R(g) \geq \lim_{Q \to \infty} \text{AveRank}(\mathcal{F}(Q)) \geq 0$.
- Spectral Theorem $\Longrightarrow$ orthonormal basis of eigenvectors of K, eigenvalues λ_j.
- $\lambda_j \geq -1$.

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Case 1: Eigenvalue (-1)

If $\exists$ a (-1)-eigenvector $f_0 \in L^2[-\frac{1}{2}, \frac{1}{2}]$ not orthogonal to 1, then $R(f_0) = \frac{\langle (I+K)f_0, f_0\rangle_{L^2}}{|\langle 1, f_0\rangle_{L^2}|^2} = \frac{\langle f_0, f_0\rangle_{L^2} - \langle f_0, f_0\rangle_{L^2}}{|\langle 1, f_0\rangle_{L^2}|^2} = 0$.

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- Fredholm Theory $\implies \exists f_0 \in L^2[-\frac{1}{2}, \frac{1}{2}]$ satisfying $(I + K)f_0 = 1$.
- $A := \langle 1, f_0 \rangle = \langle (I + K)f_0, f_0 \rangle_{L^2} > 0$.

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For $g = f_0 + h \in L^2[-\frac{1}{2}, \frac{1}{2}]$ with $\langle 1, g \rangle_{L^2} \neq 0$, WLOG $\langle 1, g \rangle_{L^2} = A$. Then $\langle 1, h \rangle_{L^2} = 0$, so

$$R(g) = \frac{\langle 1, f_0 \rangle_{L^2} + \langle (I + K)h, h \rangle_{L^2} + \langle 1, h \rangle_{L^2} + \langle h, 1 \rangle_{L^2}}{|A|^2}$$

$$= \frac{A + \langle (I + K)h, h \rangle_{L^2} + 0 + 0}{|A|^2} \geq \frac{1}{A} = R(f_0)$$

$n \geq 2$

Challenges

1. Don't have Ahiezer/Paley-Wiener to write $\hat{\phi} = g * \breve{g}$.
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A Solution

Restrict to minimizing over $\phi(x) = \sum_{j=1}^{m} \phi_{j,1}(x_1) \cdots \phi_{j,n}(x_n)$ with $\phi_{j,k}$ as in 1-level case.

Approach 1

Outline

1. Choose $\phi_{j,1}, \dots, \phi_{j,n-1}$ nicely to obtain new weight function similar to 1-level weight after integrating first $n-1$ variables.

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1. Choose $\phi_{j,1}, \ldots, \phi_{j,n-1}$ nicely to obtain new weight function similar to 1-level weight after integrating first $n-1$ variables.
2. Use 1-level approach to minimize over $\phi_{j,n}$.

Approach 1 Example: $W_{2,\mathrm{U}}$

Problem

Minimize

$$\frac{\int_{\mathbb{R}^2} \phi_1(x_1)\phi_2(x_2) W_{2,\mathrm{U}}(x)\, dx_1\, dx_2}{\phi_1(0)\phi_2(0)} = \frac{\int_{[-1,1]^2} \hat{\phi}_1(\xi_1)\hat{\phi}_2(\xi_2)\widehat{W_{2,\mathrm{U}}}(\xi)\, d\xi_1\, d\xi_2}{\phi_1(0)\phi_2(0)}$$ over ϕ_1, ϕ_2 even, Schwartz, with $\mathrm{supp}(\phi_1), \mathrm{supp}(\phi_2) \subset [-1, 1]$.

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Problem

Minimize $\frac{\int_{\mathbb{R}^2} \phi_1(x_1)\phi_2(x_2) W_{2,\mathrm{U}}(x)\, dx_1\, dx_2}{\phi_1(0)\phi_2(0)} = \frac{\int_{[-1,1]^2} \hat{\phi}_1(\xi_1)\hat{\phi}_2(\xi_2)\widehat{W_{2,\mathrm{U}}}(\xi)\, d\xi_1\, d\xi_2}{\phi_1(0)\phi_2(0)}$ over ϕ_1, ϕ_2 even, Schwartz, with $\mathrm{supp}(\phi_1), \mathrm{supp}(\phi_2) \subset [-1, 1]$.

$\chi(x) = \chi_{[-1,1]}(x)$. A short computation:

$$W_{2,\mathrm{U}}(x) = 1 - \frac{\sin^2(\pi(x_1 - x_2))}{\pi^2(x_1 - x_2)^2}$$

$$\widehat{W_{2,\mathrm{U}}}(\xi) = \delta(\xi_1)\delta(\xi_2) + \delta(\xi_1 + \xi_2)(|\xi_1| - 1)\chi(\xi_1)$$

Approach 1 Example: $W_{2,\mathrm{U}}$

For ϕ_2 arbitrary,

$$\frac{1}{\phi_2(0)}\int_{\xi_2\in\mathbb{R}}\hat{\phi}_2(\xi_2)\widehat{W_{2,\mathrm{U}}}(\xi)\,\mathrm{d}\xi_2 = \delta(\xi_1)+\frac{\hat{\phi}_2(-\xi_1)}{\phi_2(0)}(|\xi_1|-1)\chi(\xi_1)$$

$$= \delta(\xi_1)+m(\xi_1)$$

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$$= \delta(\xi_1) + m(\xi_1)$$

Choose (for example)

$$\phi_2(y) = \sqrt{\frac{2}{\pi}} \frac{\sin(y)}{y} \implies \hat{\phi}_2(\omega) = \eta(\omega) := \begin{cases} 0 & |\omega| > 1 \\ \frac{1}{2} & |\omega| = 1 \\ 1 & |\omega| < 1 \end{cases}$$

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Then

$$m(\xi_1) = (|\xi_1| - 1)\eta(\xi_1)$$

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1-level case $\implies$ optimal ϕ_1 is

$$\phi_1(x_1) = \int_{-1}^{1} (g * \breve{g})(\xi_1) e^{2\pi i x_1 \xi_1}\, \mathrm{d}\, \xi_1$$

where $g \in L^2[-\frac{1}{2}, \frac{1}{2}]$ satisfies one of

$$0 = g(x) + \int_{y=-\frac{1}{2}}^{\frac{1}{2}} (|x-y| - 1) g(y)\, \mathrm{d}y$$

$$1 = g(x) + \int_{y=-\frac{1}{2}}^{\frac{1}{2}} (|x-y| - 1) g(y)\, \mathrm{d}y$$

Approach 2

Goal

Get better bound by minimizing over all factors of $\phi(x) = \sum_{j=1}^{m} \phi_{j,1}(x_1) \cdots \phi_{j,n}(x_n)$ simultaneously.

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1. Use similar techniques to 1-level to obtain messier version of R from 1-level case.

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1. Use similar techniques to 1-level to obtain messier version of R from 1-level case.
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Outline

1. Use similar techniques to 1-level to obtain messier version of R from 1-level case.
2. Algebraic manipulations to reduce to a less messy version of R.
3. Minimize.

Approach 2 Step 1

$\widehat{W_{n,G}}$

Convolution Theorem + induction $\Longrightarrow$
$\widehat{W_{n,G}}(\xi) = \delta(x) + \sum_{f,I} f(x) \prod_{(i,j,\epsilon)\in I} \delta(x_i + \epsilon x_j)$ with $\epsilon = \pm 1$; $(i, i, -1) \notin I$; and f bounded, real-valued, even.

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Self-adjoint operator $K : L^2([-\frac{1}{2}, \frac{1}{2}]^n) \to L^2([-\frac{1}{2}, \frac{1}{2}]^n)$

$$(Kg)(x) = \int_{[-\frac{1}{2},\frac{1}{2}]^n} m(x-y) g(y) \, \mathrm{d}y_1 \cdots \mathrm{d}y_n$$

with $m(x) = \sum_{f,I} f(x) \prod_{(i,j,\epsilon)\in I} \delta(x_i + \epsilon x_j)$.

Approach 2 Step 1

For $\phi(x) = \sum_{j=1}^{\ell} \phi_{j,1}(x_1) \cdots \phi_{j,n}(x_n)$,

$$\frac{\int_{\mathbb{R}^n} \phi(x) W_{n,G}(x)}{\phi(0)} = \frac{\sum_{j=1}^{\ell} \langle (I+K) g_j, g_j \rangle_{L^2}}{\sum_{j=1}^{\ell} |\langle 1, g_j \rangle_{L^2}|^2}$$

where $g_j(x) = g_{j,1}(x_1) \cdots g_{j,n}(x_n)$.

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where $g_j(x) = g_{j,1}(x_1)\cdots g_{j,n}(x_n)$.

Goal of Step 2

Get rid of the sums...

Approach 2 Step 2

Assume $f(x) = f_1(x_1) \cdots f_n(x_n)$, $g(x) = g_1(x_1) \cdots g_n(x_n)$, $\langle 1, f \rangle_{L^2} \neq 0$, $\langle 1, g \rangle_{L^2} \neq 0$.

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$$\frac{\langle (I+K)f, f\rangle_{L^2}}{|\langle 1, f\rangle_{L^2}|^2} \leq \frac{\langle (I+K)g, g\rangle_{L^2}}{|\langle 1, g\rangle_{L^2}|^2}$$

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$$\langle (I+K)f, f\rangle_{L^2}(|\langle 1, g\rangle_{L^2}|^2 + |\langle 1, f\rangle_{L^2}|^2) \leq (\langle (I+K)g, g\rangle_{L^2} + \langle (I+K)f, f\rangle_{L^2})|\langle 1, f\rangle_{L^2}|^2$$

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Approach 2 Step 2

More manipulations...

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For $\langle 1, f\rangle_{L^2} \neq 0$ and $\langle 1, g\rangle_{L^2} \neq 0$:

$$\frac{\langle (I+K)f, f\rangle_{L^2}}{|\langle 1, f\rangle_{L^2}|^2} \le \frac{\langle (I+K)f, f\rangle_{L^2} + \langle (I+K)g, g\rangle_{L^2}}{|\langle 1, f\rangle_{L^2}|^2 + |\langle 1, g\rangle_{L^2}|^2} \le \frac{\langle (I+K)g, g\rangle_{L^2}}{|\langle 1, g\rangle_{L^2}|^2}$$

Approach 2 Step 2

Can remove terms from sum with $\langle 1, g \rangle_{L^2} \neq 0$. What if $\langle 1, g \rangle_{L^2} = 0$?

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- Take $(g_\alpha)_{\alpha=1}^{\infty}$ sequence of the form $g_\alpha(x) = g_{\alpha,1}(x_1)\cdots g_{\alpha,n}(x_n)$ with $\langle 1, g_\alpha\rangle_{L^2} \neq 0$ and $g_\alpha \to g$ in L^2.

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Reduced Problem

Suffices to minimize $R(g) := \frac{\langle (I+K)g,g\rangle_{L^2}}{|\langle 1,g\rangle_{L^2}|^2}$ over $g(x) = g_1(x_1)\cdots g_n(x_n)$ with $\langle 1, g\rangle_{L^2} \neq 0$.

Approach 2 Step 3

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More careful treatment of K:

- Working termwise in $L^2([-\frac{1}{2}, \frac{1}{2}]^k)$ for $k < n$.

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- More explicit calculation of $\widehat{W_{n,G}}$?

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