Zeros of L-functions near the Central Point and Optimal Test Functions

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  - ◇ Riemann zeta function, general *L*-functions, RMT
  - ◇ nuclear physics, Birch and Swinnerton Dyer conjecture, elliptic curve cryptography.
  - ◇ *n*-level densities, Katz-Sarnak determinants

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  - ◇ n-level densities, Katz-Sarnak determinants

- Bounding the average rank:
  - ◇ Saturated bound for 1-dimensional case.
  - ◇ Extending to give a strong bound for $n \geq 2$ dimensions.
  - ◇ Example with $W_{2,\mathrm{U}}$ and numerical data.

Example: Riemann Zeta Function

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \text{ for } \Re(s) > 1.$$

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Functional Equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \text{ for } s \in \mathbb{C} \setminus \{1\}.$$

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Riemann Hypothesis

All nontrivial zeros (not negative even integers) of ζ are of the form $\gamma = \frac{1}{2} + i\sigma$ with $\sigma \in \mathbb{R}$.

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- meromorphic continuation to $\mathbb{C}$, of finite order, at most finitely may poles (all on the line $\Re(s) = 1$).
- Functional equation: $\omega \in \mathbb{R}$, $G(s)$ product of Γ-fns:

$$e^{i\omega} G(s) L(s, f) = e^{-i\omega} \overline{G(1 - \overline{s}) L(1 - \overline{s})}.$$

Applications

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- zeros $\longleftrightarrow$ primes.
- Birch and Swinnerton-Dyer conjecture.
  - $\diamond$ Elliptic curve cryptography.

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- Ensembles of matrices (e.g. real symmetric, Hermitian) with entries drawn from probability distribution.
- Study distribution of normalized eigenvalues for given ensemble.

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Applications

- Energy levels of heavy nuclei.
- Elliptic curve cryptography.

1-level Density

Family of Dirichlet L-functions $L(s, f)$ indexed by cuspidal newform $f \in \mathcal{F}$. Riemann hypothesis $\implies$ zeros of $L(s, f)$ are of the form $\rho_f = \frac{1}{2} + \mathrm{i}\gamma_f$ with $\gamma_f \in \mathbb{R}$.

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1-level Density

$D(f;\phi) := \sum_{\gamma_f} \phi(\frac{\gamma_f}{2\pi}\log(c_f))$ where $\phi \geq 0$ is even, Schwartz,

Fourier transform $\hat{\phi}$ compactly supported, $\phi(0) > 0$.

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Idea:

Varying ϕ, $D(f; \phi)$ measures density of zeros of $L(s, f)$ near central point $s = \frac{1}{2}$.

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Then take a limit:

$$\lim_{Q \to \infty} \mathbb{E}(\mathcal{F}(Q);\phi) = \int_{-\infty}^{\infty} \phi(x) W(\mathcal{F})(x)\,\mathrm{d}x$$

where $W(\mathcal{F})$ is a distribution depending on $\mathcal{F}$.

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From RMT, $W(\mathcal{F})$ is dependent on a symmetry group $G = G(\mathcal{F})$ of $\mathcal{F}$. Write $W(\mathcal{F}) = W_{1,G}$. Some examples:

$$W_{1,\mathrm{O}}(x) = 1 + \frac{1}{2}\delta(x)$$

$$W_{1,\mathrm{SO(Even)}}(x) = 1 + \frac{\sin(2\pi x)}{2\pi x}$$

$$W_{1,\mathrm{SO(Odd)}}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x} + \delta(x)$$

1-level Density

Quantity of interest

$\lim_{Q\to\infty}$ AveRank($\mathcal{F}(Q)$), where AveRank($\mathcal{F}(Q)$) is average order of vanishing of the L-functions with $f \in \mathcal{F}(Q)$ at $s = 1/2$.

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Can show that

$$\lim_{Q\to\infty} \text{AveRank}(\mathcal{F}(Q)) \leq \frac{\int_{-\infty}^{\infty} \phi(x) W_{1,G}(x)\, \mathrm{d}x}{\phi(0)}$$

n-level Density

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$$D_n(f;\phi) := \sum_{\substack{\gamma_{j,f} \\ |j| \text{ distinct}}} \phi\left(\frac{\gamma_{1,f}}{2\pi}\log(c_f), \frac{\gamma_{2,f}}{2\pi}\log(c_f), \ldots, \frac{\gamma_{n,f}}{2\pi}\log(c_f)\right)$$

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Higher Dimensional Bound

$$\lim_{Q\to\infty} \text{WeightedAveRank}(\mathcal{F}(Q)) \le \frac{\int_{\mathbb{R}^n} \phi(x) W_{n,G}(x)\, dx_1 \cdots dx_n}{\phi(0)}$$

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Goal

Higher level densities give stronger bound. Minimize right-hand side over admissible ϕ for n as large as possible.

Katz-Sarnak Determinants

Set $K_\epsilon(x,y) := \frac{\sin(\pi(x-y))}{\pi(x-y)} + \epsilon\frac{\sin(\pi(x+y))}{\pi(x+y)}$, $\epsilon \in \{0, \pm 1\}$. n-level weights for classical compact groups are:

$$W_{n,\mathrm{SO(Even)}}(x) = \det\left(K_1(x_i,x_j)\right)_{i,j\le n}$$

$$W_{n,\mathrm{SO(Odd)}}(x) = \det\left(K_{-1}(x_i,x_j)\right)_{i,j\le n} + \sum_{k=1}^{n} \delta(x_k)\det\left(K_{-1}(x_i,x_j)\right)_{i,j\ne k}$$

$$W_{n,\mathrm{O}}(x) = \frac{1}{2}W_{n,\mathrm{SO(Even)}}(x) + \frac{1}{2}W_{n,\mathrm{SO(Odd)}}(x)$$

$$W_{n,\mathrm{U}}(x) = \det\left(K_0(x_i,x_j)\right)_{i,j\le n}$$

$$W_{n,\mathrm{Sp}}(x) = \det\left(K_{-1}(x_i,x_j)\right)_{i,j\le n}$$

Main Results

Main Idea

Restrict domain to only those ϕ which are products of single variable test functions: $\phi(x) = \phi_1(x_1) \cdots \phi_n(x_n)$ (equivalent to linear combinations of such products).

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Main Result 1

1. Choosing first $n-1$ factors $\phi_1, \dots, \phi_{n-1}$ carefully, can integrate first $n-1$ variables to obtain new weight function of a form similar to 1-dimensional weights.
2. 1-level case already solved, so choose ϕ_n optimally for new weight.

Main Results

Main Idea

Restrict domain to only those ϕ which are products of single variable test functions: $\phi(x) = \phi_1(x_1) \cdots \phi_n(x_n)$.

Main Result 2

1. Using functional analysis, reduce to a problem similar to the 1-level case.
2. Deal with the added difficulties of poorly behaved Fourier transforms of weight functions.

1-level Case

2 Steps

1. Reduce problem to different optimization problem.

1-level Case

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1. Reduce problem to different optimization problem.
2. Use functional analysis to solve reduced problem.

Step 1: Reduce Problem

Assume $\text{supp}(\hat{\phi}) \subset [-1, 1]$. Plancherel on numerator, taking then inverting Fourier transform in denominator:

$$\frac{\int_{-\infty}^{\infty} \phi(x) W_{1,G}(x)\, \mathrm{d}x}{\phi(0)} = \frac{\int_{-1}^{1} \hat{\phi}(\xi) \widehat{W_{1,G}}(\xi)\, \mathrm{d}\xi}{\int_{-1}^{1} \hat{\phi}(\xi)\, \mathrm{d}\xi}.$$

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Ahiezer's Theorem and the Paley-Wiener Theorem show ϕ admissible $\iff$ $\hat{\phi}(\xi) = (g * \breve{g})(\xi)$ for some $g \in L^2[-\frac{1}{2}, \frac{1}{2}]$, where $\breve{g}(\xi) = \overline{g(-\xi)}$.

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Some calculations show for $G =$ classical compact group, $\widehat{W_{1,G}}(\xi) = \delta(\xi) + m(\xi)$ on $[-1, 1]$, with $m(\xi)$ real, piecewise continuous, even.

Step 1: Reduce Problem

Some functional analysis: define compact, self-adjoint linear operator $K : L^2[-\frac{1}{2}, \frac{1}{2}] \to L^2[-\frac{1}{2}, \frac{1}{2}]$

$$(Kg)(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} m(x-y)g(y)\,\mathrm{d}y.$$

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Some manipulations:

$$\frac{\int_{-1}^{1} \hat{\phi}(\xi)\widehat{W_{1,G}}(\xi)\,\mathrm{d}\xi}{\int_{-1}^{1} \hat{\phi}(\xi)\,\mathrm{d}\xi} = \frac{\int_{-1}^{1}(g * \breve{g})(\xi)(\delta(\xi) + m(\xi))\,\mathrm{d}\xi}{\int_{-1}^{1}(g * \breve{g})(\xi)\,\mathrm{d}\xi}$$

Step 1: Reduce Problem

$$= \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{1} \left(\delta(\xi) g(\xi+y)\overline{g(y)} + m(\xi) g(\xi+y)\overline{g(y)}\right) \, \mathrm{d}\xi \, \mathrm{d}y}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{1} g(\xi+y)\overline{g(y)} \, \mathrm{d}\xi \, \mathrm{d}y}$$

$$= \frac{\langle g, g\rangle_{L^2} + \int_{-1}^{1} \int_{-\frac{1}{2}+\xi}^{\frac{1}{2}+\xi} m(\xi) g(y) \overline{g(-\xi+y)} \, \mathrm{d}y \, \mathrm{d}\xi}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{1} g(\xi+y) \, \mathrm{d}\xi \, \overline{g(y)} \, \mathrm{d}y}$$

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Step 1: Reduce Problem

$$
\begin{aligned}
&= \frac{\langle g, g\rangle_{L^2} + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} m(\xi - y) g(y)\, \mathrm{d}y \overline{g(\xi)}\, \mathrm{d}\xi}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{1} g(\xi + y)\, \mathrm{d}\xi\, \overline{g(y)}\, \mathrm{d}y} \\
&= \frac{\langle g, g\rangle_{L^2} + \langle Kg, g\rangle_{L^2}}{\langle g, \mathbf{1}\rangle_{L^2} \langle \mathbf{1}, g\rangle_{L^2}} \\
&= \frac{\langle (I + K)g, g\rangle_{L^2}}{|\langle \mathbf{1}, g\rangle_{L^2}|^2}.
\end{aligned}
$$

1 is characteristic function of appropriate set.

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$$= \frac{\langle g, g\rangle_{L^2} + \langle Kg, g\rangle_{L^2}}{\langle g, \mathbf{1}\rangle_{L^2}\langle \mathbf{1}, g\rangle_{L^2}}$$

$$= \frac{\langle (I+K)g, g\rangle_{L^2}}{|\langle \mathbf{1}, g\rangle_{L^2}|^2}.$$

$\mathbf{1}$ is characteristic function of appropriate set.

New Problem

Defining $R : L^2[-\frac{1}{2}, \frac{1}{2}] \to L^2[-\frac{1}{2}, \frac{1}{2}]$ by $R(g) := \frac{\langle (I+K)g, g\rangle_{L^2}}{|\langle \mathbf{1}, g\rangle_{L^2}|^2}$, minimize R over subset of $L^2[-\frac{1}{2}, \frac{1}{2}]$ with denominator $\neq 0$.

Step 2: Minimization

Some observations:

- $R(g) \geq \lim_{Q\to\infty} \mathrm{AveRank}(\mathcal{F}(Q)) \geq 0$.
- Spectral Theorem $\implies$ orthonormal basis of eigenvectors of K, eigenvalues λ_j.
- $\lambda_j \geq -1$.

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Case 1: Eigenvalue (-1)

If $\exists$ a (-1)-eigenvector $f_0 \in L^2[-\frac{1}{2}, \frac{1}{2}]$ not orthogonal to 1, then $R(f_0) = \frac{\langle (I+K)f_0, f_0\rangle_{L^2}}{|\langle \mathbf{1}, f_0\rangle_{L^2}|^2} = \frac{\langle f_0, f_0\rangle_{L^2} - \langle f_0, f_0\rangle_{L^2}}{|\langle \mathbf{1}, f_0\rangle_{L^2}|^2} = 0$.

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- $\ker(I + K) = \{0\}$ (all eigenvalues > -1).
- Fredholm Theory $\Longrightarrow$ $\exists f_0 \in L^2[-\frac{1}{2}, \frac{1}{2}]$ satisfying $(I + K)f_0 = \mathbf{1}$.
- $A := \langle \mathbf{1}, f_0 \rangle = \langle (I + K)f_0, f_0 \rangle_{L^2} > 0$.

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For $g = f_0 + h \in L^2[-\frac{1}{2},\frac{1}{2}]$ with $\langle \mathbf{1}, g\rangle_{L^2} \neq 0$, WLOG $\langle \mathbf{1}, g\rangle_{L^2} = A$. Then $\langle \mathbf{1}, h\rangle_{L^2} = 0$, so

$$R(g) = \frac{\langle \mathbf{1}, f_0\rangle_{L^2} + \langle (I+K)h, h\rangle_{L^2} + \langle \mathbf{1}, h\rangle_{L^2} + \langle h, \mathbf{1}\rangle_{L^2}}{|A|^2}$$

$$= \frac{A + \langle (I+K)h, h\rangle_{L^2} + 0 + 0}{|A|^2} \geq \frac{1}{A} = R(f_0)$$

$n \geq 2$

Challenges:

1. $\widehat{W_{n,G}}$ more complicated.
2. Higher dimensional integral operators not as well-understood.

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A Solution

Restrict to minimizing over $\phi(x) = \phi_1(x_1)\cdots\phi_n(x_n)$ with ϕ_j as in 1-level case (equivalent to minimizing over finite sums).

Approach 1

Outline

1. Choose $\phi_2, \ldots, \phi_n$ and integrate last $n-1$ variables to obtain new weight function similar to 1-level weights.

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1. Choose $\phi_2, \dots, \phi_n$ and integrate last $n - 1$ variables to obtain new weight function similar to 1-level weights.
2. Use 1-level approach to minimize choice of ϕ_1.

Approach 1 Example: $W_{2,\mathrm{U}}$

Problem

Minimize

$$\frac{\int_{\mathbb{R}^2} \phi_1(x_1)\phi_2(x_2)W_{2,\mathrm{U}}(x)\,\mathrm{d}x_1\,\mathrm{d}x_2}{\phi_1(0)\phi_2(0)} = \frac{\int_{[-1,1]^2} \hat{\phi}_1(\xi_1)\hat{\phi}_2(\xi_2)\widehat{W_{2,\mathrm{U}}}(\xi)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2}{\phi_1(0)\phi_2(0)}$$ over ϕ_1, ϕ_2 even, Schwartz, $\phi_1(0), \phi_2(0) > 0$, and $\mathrm{supp}(\hat{\phi}_1), \mathrm{supp}(\hat{\phi}_2) \subset [-1, 1]$.

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ϕ_1, ϕ_2 even, Schwartz, $\phi_1(0), \phi_2(0) > 0$, and $\mathrm{supp}(\hat{\phi}_1), \mathrm{supp}(\hat{\phi}_2) \subset [-1, 1]$.

$\mathbf{1}(x)$ characteristic function of appropriate set. A short computation:

$$W_{2,\mathrm{U}}(x) = 1 - \frac{\sin^2(\pi(x_1 - x_2))}{\pi^2(x_1 - x_2)^2}$$

$$\widehat{W_{2,\mathrm{U}}}(\xi) = \delta(\xi_1)\delta(\xi_2) + \delta(\xi_1 + \xi_2)(|\xi_1| - 1)\mathbf{1}(\xi_1)$$

Approach 1 Example: $W_{2,\mathrm{U}}$

For ϕ_2 arbitrary,

$$\frac{1}{\phi_2(0)} \int_{\xi_2 \in \mathbb{R}} \hat{\phi}_2(\xi_2) \widehat{W_{2,\mathrm{U}}}(\xi) \, \mathrm{d}\xi_2 = \frac{\hat{\phi}_2(0)}{\phi_2(0)} \delta(\xi_1) + \frac{\hat{\phi}_2(-\xi_1)}{\phi_2(0)} (|\xi_1| - 1) \mathbf{1}(\xi_1)$$

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New Problem:

Normalizing by $\frac{\hat{\phi}_2(0)}{\phi_2(0)}$, minimize

$$\frac{\int_{\xi_1 \in \mathbb{R}} \hat{\phi}_1(\xi_1) \widetilde{W}(\xi_1)}{\phi_1(0)}$$

over ϕ_1, where $\widetilde{W}(\xi_1) = \delta(\xi_1) + \frac{\hat{\phi}_2(-\xi_1)}{\hat{\phi}_2(0)} (|\xi_1| - 1)\mathbf{1}(\xi_1)$.

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$$\widetilde{W}(\xi_1) = \delta(\xi_1) + \frac{\hat{\phi}_2(-\xi_1)}{\hat{\phi}_2(0)}(|\xi_1| - 1)\mathbf{1}(x)(\xi_1) = \delta(\xi_1) + m(\xi_1)$$

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- ϕ_2 even $\implies$ m is even.
- 1-level case $\implies$ optimal ϕ_1 has $\hat{\phi}_1(\xi_1) = (g * \breve{g})(\xi_1)$ where $g \in L^2[-\frac{1}{2}, \frac{1}{2}]$ satisfying

$$\mathbf{1}(x) = g(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} m(x-y)g(y)\,\mathrm{d}y.$$

Minimum value is $\frac{1}{\langle \mathbf{1}, g\rangle_{L^2}}$.

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- $g(x) = \mathbf{1}(x) + \sum_{n=1}^{\infty} K_n(x)$.
- $\langle \mathbf{1}, g\rangle_{L^2} = 1 + \sum_{n=1}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} K_n(x)\,\mathrm{d}x$.

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- Terms of series are nonnegative, so truncate after finitely many terms to get

$$\frac{\hat{\phi}_2(0)}{\phi_2(0)} \frac{1}{\langle \mathbf{1}, g\rangle_{L^2}} \leq \frac{\hat{\phi}_2(0)}{\phi_2(0)} \left(1 + \sum_{n=1}^{3} \int_{-\frac{1}{2}}^{\frac{1}{2}} K_n(x)\,\mathrm{d}x\right)^{-1} \approx .519105$$

Numerical Data for $n = 2$ (Truncate at 3 terms)

	$h(y)$	Bound
$W_{2,\mathrm{O}}$	$(1-\lvert y\rvert)\mathbf{1}(y)$	0.290701
$W_{2,\mathrm{SO(Even)}}$	$\mathbf{1}(y)$	0.371402
$W_{2,\mathrm{SO(Odd)}}$	$\mathbf{1}(y)$	0.447178
$W_{2,\mathrm{U}}$	$(1-\lvert y\rvert)\mathbf{1}(y)$	0.519105
$W_{2,\mathrm{Sp}}$	$\mathbf{1}(y)$	0.447178

Red = numerical approximation up to small error.

Applications to Order of Vanishing

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$$\Pr(0)+\Pr(1) \geq \begin{cases} 0.709299 & W_{2,\mathrm{O}} \\ 0.628598 & W_{2,\mathrm{SO(Even)}} \\ 0.552822 & W_{2,\mathrm{SO(Odd)}} \\ 0.480895 & W_{2,\mathrm{U}} \\ 0.552822 & W_{2,\mathrm{Sp}} \end{cases}$$

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Odd:

$$\Pr(1) \geq \begin{cases} 0.951550 & W_{2,\mathrm{O}} \\ 0.938100 & W_{2,\mathrm{SO(Even)}} \\ 0.925470 & W_{2,\mathrm{SO(Odd)}} \\ 0.913483 & W_{2,\mathrm{U}} \\ 0.925470 & W_{2,\mathrm{Sp}} \end{cases}$$

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