Zeros of $L$-functions near the Central Point and Optimal Test Functions

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Summary

- Review of $L$-functions:
  - Riemann zeta function, general $L$-functions, RMT
  - Nuclear physics, Birch and Swinnerton Dyer conjecture, elliptic curve cryptography.
  - $n$-level densities, Katz-Sarnak determinants

Bounding the average rank:
  - Saturated bound for 1-dimensional case.
  - Extending to give a strong bound for $n \geq 2$ dimensions.
  - Example with $W_2$, $U$ and numerical data.
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- Example with $W_{2,U}$ and numerical data.
Example: Riemann Zeta Function

Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \text{ for } \Re(s) > 1. \]
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Functional Equation

\[ \zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1 - s) \zeta(1 - s) \text{ for } s \in \mathbb{C} \setminus \{1\}. \]
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Riemann Hypothesis

All nontrivial zeros (not negative even integers) of \( \zeta \) are of the form \( \gamma = \frac{1}{2} + i\sigma \) with \( \sigma \in \mathbb{R} \).
A Dirichlet $L$-function $L(s, f)$ with cuspidal newform $f$ satisfies:
General $L$-functions

A Dirichlet $L$-function $L(s, f)$ with cuspidal newform $f$ satisfies:

- Euler product:

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{\text{prime } p} \prod_{j=1}^{d} \left(1 - \alpha_{f,j}(p)p^{-s}\right)^{-1}.$$
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- meromorphic continuation to $\mathbb{C}$, of finite order, at most finitely may poles (all on the line $\Re(s) = 1$).
General L-functions

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- meromorphic continuation to $\mathbb{C}$, of finite order, at most finitely may poles (all on the line $\Re(s) = 1$).

- Functional equation: $\omega \in \mathbb{R}$, $G(s)$ product of $\Gamma$-fns:

$$e^{i\omega} G(s)L(s, f) = e^{-i\omega} G(1 - \overline{s})L(1 - \overline{s}).$$
Applications

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- zeros $\leftrightarrow$ primes.
- Birch and Swinnerton-Dyer conjecture.
  - Elliptic curve cryptography.
Random Matrix Theory (RMT)

- Ensembles of matrices (e.g. real symmetric, Hermitian) with entries drawn from probability distribution.
- Study distribution of normalized eigenvalues for given ensemble.
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Number Theory and RMT
Behavior of zeros of $L$-functions well-modeled by RMT models.
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- Study distribution of normalized eigenvalues for given ensemble.

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Behavior of zeros of $L$-functions well-modeled by RMT models.

Applications
- Energy levels of heavy nuclei.
- Elliptic curve cryptography.
1-level Density

Family of Dirichlet $L$-functions $L(s, f)$ indexed by cuspidal newform $f \in \mathcal{F}$. Riemann hypothesis $\implies$ zeros of $L(s, f)$ are of the form $\rho_f = \frac{1}{2} + i \gamma_f$ with $\gamma_f \in \mathbb{R}$.
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1-level Density

$$D(f; \phi) := \sum_{\gamma_f} \phi\left(\frac{\gamma_f}{2\pi} \log(c_f)\right)$$

where $\phi \geq 0$ is even, Schwartz, Fourier transform $\hat{\phi}$ compactly supported, $\phi(0) > 0$. $c_f > 1$ is the analytic conductor.
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Idea:

Varying $\phi$, $D(f; \phi)$ measures density of zeros of $L(s, f)$ near central point $s = \frac{1}{2}$. 
1-level Density

Impossible to calculate $D(f; \phi)$ explicitly in practice...
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Impossible to calculate $D(f; \phi)$ explicitly in practice… so take averages over finite subfamilies of $\mathcal{F}$:

$$\mathcal{F}(Q) := \{ f \in \mathcal{F} : c_f \leq Q \}$$

$$\mathbb{E}(\mathcal{F}(Q); \phi) := \frac{1}{|\mathcal{F}(Q)|} \sum_{f \in \mathcal{F}} D(f; \phi)$$
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Then take a limit:

$$\lim_{Q \to \infty} \mathbb{E}(\mathcal{F}(Q); \phi) = \int_{-\infty}^{\infty} \phi(x) W(\mathcal{F})(x) \, dx$$

where $W(\mathcal{F})$ is a distribution depending on $\mathcal{F}$. 
1-level Density

From RMT, $W(\mathcal{F})$ is dependent on a symmetry group $G = G(\mathcal{F})$ of $\mathcal{F}$. Write $W(\mathcal{F}) = W_{1,G}$. 
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$$W_{1,O}(x) = 1 + \frac{1}{2} \delta(x)$$

$$W_{1,SO(\text{Even})}(x) = 1 + \frac{\sin(2\pi x)}{2\pi x}$$

$$W_{1,SO(\text{Odd})}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x} + \delta(x)$$
1-level Density

Quantity of interest

$$\lim_{Q \to \infty} \text{AveRank}(\mathcal{F}(Q))$$, where $\text{AveRank}(\mathcal{F}(Q))$ is average order of vanishing of the $L$-functions with $f \in \mathcal{F}(Q)$ at $s = 1/2$. 

Can show that

$$\lim_{Q \to \infty} \text{AveRank}(\mathcal{F}(Q)) \leq \int_{-\infty}^{\infty} \phi(x) W_1 G(x) \, dx \phi(0)$$
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Can show that

$$\lim_{Q \to \infty} \text{AveRank}(\mathcal{F}(Q)) \leq \frac{\int_{-\infty}^{\infty} \phi(x) W_{1,G}(x) \, dx}{\phi(0)}$$
\( D_n(f; \phi) := \sum_{|j| \text{ distinct}} \phi \left( \frac{\gamma_{1,f}}{2\pi} \log(c_f), \frac{\gamma_{2,f}}{2\pi} \log(c_f), \ldots, \frac{\gamma_{n,f}}{2\pi} \log(c_f) \right) \)
\( D_n(f; \phi) := \sum_{\gamma_1, j, f} \phi \left( \frac{\gamma_1, f}{2\pi} \log(c_f), \frac{\gamma_2, f}{2\pi} \log(c_f), \ldots, \frac{\gamma_n, f}{2\pi} \log(c_f) \right) \) for \(|j| \) distinct.

\[
\lim_{Q \to \infty} \text{WeightedAveRank}(\mathcal{F}(Q)) \leq \frac{\int_{\mathbb{R}^n} \phi(x) W_{n,G}(x) \, dx_1 \cdots dx_n}{\phi(0)}
\]
**$n$-level Density**

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$$D_n(f; \phi) := \sum_{\gamma_{j,f} \text{ distinct}} \phi \left( \frac{\gamma_{1,f}}{2\pi} \log(c_f), \frac{\gamma_{2,f}}{2\pi} \log(c_f), \ldots, \frac{\gamma_{n,f}}{2\pi} \log(c_f) \right)$$

**Higher Dimensional Bound**

$$\lim_{Q \to \infty} \text{WeightedAveRank}(\mathcal{F}(Q)) \leq \frac{\int_{\mathbb{R}^n} \phi(x) W_{n,G}(x) \, dx_1 \ldots dx_n}{\phi(0)}$$

**Goal**

Higher level densities give stronger bound. Minimize right-hand side over admissible $\phi$ for $n$ as large as possible.
Katz-Sarnak Determinants

Set $K_\epsilon(x, y) := \frac{\sin(\pi(x-y))}{\pi(x-y)} + \epsilon \frac{\sin(\pi(x+y))}{\pi(x+y)}$, $\epsilon \in \{0, \pm 1\}$. $n$-level weights for classical compact groups are:

$W_{n,SO(\text{Even})}(x) = \det (K_1(x_i, x_j))_{i,j \leq n}$

$W_{n,SO(\text{Odd})}(x) = \det (K_{-1}(x_i, x_j))_{i,j \leq n} + \sum_{k=1}^{n} \delta(x_k) \det (K_{-1}(x_i, x_j))_{i,j \neq k}$

$W_{n,O}(x) = \frac{1}{2} W_{n,SO(\text{Even})}(x) + \frac{1}{2} W_{n,SO(\text{Odd})}(x)$

$W_{n,U}(x) = \det (K_0(x_i, x_j))_{i,j \leq n}$

$W_{n,Sp}(x) = \det (K_{-1}(x_i, x_j))_{i,j \leq n}$
Main Results

Main Idea

Restrict domain to only those $\phi$ which are products of single variable test functions: $\phi(x) = \phi_1(x_1) \cdots \phi_n(x_n)$ (equivalent to linear combinations of such products).
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Main Result 1
1. Choosing first $n - 1$ factors $\phi_1, \ldots, \phi_{n-1}$ carefully, can integrate first $n - 1$ variables to obtain new weight function of a form similar to 1-dimensional weights.
2. 1-level case already solved, so choose $\phi_n$ optimally for new weight.
Main Results

Main Idea
Restrict domain to only those $\phi$ which are products of single variable test functions: $\phi(x) = \phi_1(x_1) \cdots \phi_n(x_n)$.

Main Result 2
1. Using functional analysis, reduce to a problem similar to the 1-level case.
2. Deal with the added difficulties of poorly behaved Fourier transforms of weight functions.
1-level Case

2 Steps

1. Reduce problem to different optimization problem.
## 1-level Case

### 2 Steps

1. Reduce problem to different optimization problem.
2. Use functional analysis to solve reduced problem.
Step 1: Reduce Problem

Assume \( \text{supp}(\hat{\phi}) \subset [-1, 1] \). Plancherel on numerator, taking then inverting Fourier transform in denominator:

\[
\frac{\int_{-\infty}^{\infty} \phi(x) W_{1,G}(x) \, dx}{\phi(0)} = \frac{\int_{-1}^{1} \hat{\phi}(\xi) \hat{W}_{1,G}(\xi) \, d\xi}{\int_{-1}^{1} \hat{\phi}(\xi) \, d\xi}.
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Ahiezer’s Theorem and the Paley-Wiener Theorem show \( \phi \) admissible \( \iff \hat{\phi}(\xi) = (g \ast \check{g})(\xi) \) for some \( g \in L^2\left[-\frac{1}{2}, \frac{1}{2}\right] \), where \( \check{g}(\xi) = g(-\xi) \).
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Some calculations show for \( G = \) classical compact group, \( \hat{W}_{1,G}(\xi) = \delta(\xi) + m(\xi) \) on \([-1, 1]\), with \( m(\xi) \) real, piecewise continuous, even.
Step 1: Reduce Problem

Some functional analysis: define compact, self-adjoint linear operator $K : L^2[-\frac{1}{2}, \frac{1}{2}] \rightarrow L^2[-\frac{1}{2}, \frac{1}{2}]$

$$(Kg)(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} m(x - y)g(y) \, dy.$$
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Some manipulations:

$$\frac{\int_{-1}^{1} \hat{\phi}(\xi) \overline{W_{1, G}(\xi)} \, d\xi}{\int_{-1}^{1} \hat{\phi}(\xi) \, d\xi} = \frac{\int_{-1}^{1} (g \ast \tilde{g})(\xi)(\delta(\xi) + m(\xi)) \, d\xi}{\int_{-1}^{1} (g \ast \tilde{g})(\xi) \, d\xi}$$
Step 1: Reduce Problem

\[
\int_{-1/2}^{1/2} \int_{-1}^{1} \left( \delta(\xi) g(\xi + y) \overline{g(y)} + m(\xi) g(\xi + y) \overline{g(y)} \right) \, d\xi \, dy \\
= \frac{\int_{-1/2}^{1/2} \int_{-1}^{1} g(\xi + y) \overline{g(y)} \, d\xi \, dy}{\int_{-1/2}^{1/2} \int_{-1}^{1} g(\xi + y) \overline{g(y)} \, d\xi \, dy}
\]

\[
\langle g, g \rangle_{L^2} + \int_{-1}^{1} \int_{-1/2}^{1/2 + \xi} m(\xi) g(y) \overline{g(-\xi + y)} \, dy \, d\xi
\]

\[
= \frac{\int_{-1/2}^{1/2} \int_{-1}^{1} g(\xi + y) \, d\xi \overline{g(y)} \, dy}{\int_{-1/2}^{1/2} \int_{-1}^{1} g(\xi + y) \, d\xi \overline{g(y)} \, dy}
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\]
\[
= \frac{\langle g, g \rangle_{L^2} + \langle Kg, g \rangle_{L^2}}{\langle g, 1 \rangle_{L^2} \langle 1, g \rangle_{L^2}}
\]
\[
= \frac{\langle (I + K)g, g \rangle_{L^2}}{|\langle 1, g \rangle_{L^2}|^2}.
\]

1 is characteristic function of appropriate set.
Step 1: Reduce Problem

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New Problem

Defining \( R : L^2[-\frac{1}{2}, \frac{1}{2}] \to L^2[-\frac{1}{2}, \frac{1}{2}] \) by \( R(g) := \frac{\langle (I + K)g, g \rangle_{L^2}}{|\langle 1, g \rangle_{L^2}|^2} \), minimize \( R \) over subset of \( L^2[-\frac{1}{2}, \frac{1}{2}] \) with denominator \( \neq 0 \).
Step 2: Minimization

Some observations:

- \( R(g) \geq \lim_{Q \to \infty} \text{AveRank}(F(Q)) \geq 0. \)
- Spectral Theorem \( \iff \) orthonormal basis of eigenvectors of \( K \), eigenvalues \( \lambda_j \).
- \( \lambda_j \geq -1. \)
Step 2: Minimization

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- \( R(g) \geq \lim_{Q \to \infty} \text{AveRank}(\mathcal{F}(Q)) \geq 0. \)

- Spectral Theorem \( \implies \) orthonormal basis of eigenvectors of \( K \), eigenvalues \( \lambda_j \).

- \( \lambda_j \geq -1. \)

**Case 1: Eigenvalue \((-1)\)**

If \( \exists \) a \((-1)\)-eigenvector \( f_0 \in L^2[-\frac{1}{2}, \frac{1}{2}] \) not orthogonal to 1, then \( R(f_0) = \frac{\langle (I+K)f_0, f_0 \rangle_{L^2}}{|\langle 1, f_0 \rangle_{L^2}|^2} = \frac{\langle f_0, f_0 \rangle_{L^2} - \langle f_0, f_0 \rangle_{L^2}}{|\langle 1, f_0 \rangle_{L^2}|^2} = 0. \)
Step 2: Minimization

Case 2: $\lambda_j > -1$ for all $j$. More functional analysis!
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- $\ker(I + K) = \{0\}$ (all eigenvalues $> -1$).
- Fredholm Theory $\implies \exists f_0 \in L^2[-\frac{1}{2}, \frac{1}{2}]$ satisfying $(I + K)f_0 = 1$.
- $A := \langle 1, f_0 \rangle = \langle (I + K)f_0, f_0 \rangle_{L^2} > 0$. 
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- $A := \langle 1, f_0 \rangle = \langle (I + K)f_0, f_0 \rangle_{L^2} > 0$.

For $g = f_0 + h \in L^2[-\frac{1}{2}, \frac{1}{2}]$ with $\langle 1, g \rangle_{L^2} \neq 0$, WLOG $\langle 1, g \rangle_{L^2} = A$. Then $\langle 1, h \rangle_{L^2} = 0$, so

$$R(g) = \frac{\langle 1, f_0 \rangle_{L^2} + \langle (I + K)h, h \rangle_{L^2} + \langle 1, h \rangle_{L^2} + \langle h, 1 \rangle_{L^2}}{|A|^2}$$

$$= \frac{A + \langle (I + K)h, h \rangle_{L^2} + 0 + 0}{|A|^2} \geq \frac{1}{A} = R(f_0)$$
$n \geq 2$

**Challenges:**

1. $\widehat{\mathcal{W}}_{n,G}$ more complicated.
2. Higher dimensional integral operators not as well-understood.
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1. \( \mathcal{W}_{n,G} \) more complicated.
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A Solution

Restrict to minimizing over \( \phi(x) = \phi_1(x_1) \cdots \phi_n(x_n) \) with \( \phi_j \) as in 1-level case (equivalent to minimizing over finite sums).
Approach 1

Outline

1. Choose $\phi_2, \ldots, \phi_n$ and integrate last $n - 1$ variables to obtain new weight function similar to 1-level weights.
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1. Choose $\phi_2, \ldots, \phi_n$ and integrate last $n - 1$ variables to obtain new weight function similar to 1-level weights.
2. Use 1-level approach to minimize choice of $\phi_1$. 
Approach 1 Example: $W_{2,U}$

**Problem**

Minimize

$$\int_{\mathbb{R}^2} \frac{\phi_1(x_1)\phi_2(x_2) W_{2,U}(x)}{\phi_1(0)\phi_2(0)} \, dx_1 \, dx_2 = \frac{\int_{[-1,1]^2} \hat{\phi}_1(\xi_1)\hat{\phi}_2(\xi_2) \hat{W}_{2,U}(\xi) \, d\xi_1 \, d\xi_2}{\phi_1(0)\phi_2(0)}$$

over $\phi_1, \phi_2$ even, Schwartz, $\phi_1(0), \phi_2(0) > 0$, and $\text{supp}(\hat{\phi}_1), \text{supp}(\hat{\phi}_2) \subset [-1, 1]$. 

1. Approach

Example: $W_{2,U}$
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$1(x)$ characteristic function of appropriate set. A short computation:

$$
W_{2,U}(x) = 1 - \frac{\sin^2(\pi(x_1 - x_2))}{\pi^2(x_1 - x_2)^2}
$$

$$
\hat{W}_{2,U}(\xi) = \delta(\xi_1)\delta(\xi_2) + \delta(\xi_1 + \xi_2)(|\xi_1| - 1)1(\xi_1)
$$
Approach 1 Example: $W_{2,U}$

For $\phi_2$ arbitrary,

$$\frac{1}{\phi_2(0)} \int_{\xi_2 \in \mathbb{R}} \hat{\phi}_2(\xi_2) \hat{W}_{2,U}(\xi) \, d\xi_2 = \frac{\phi_2(0)}{\phi_2(0)} \delta(\xi_1) + \frac{\phi_2(-\xi_1)}{\phi_2(0)} (|\xi_1| - 1) 1(\xi_1)$$
Approach 1 Example: $W_{2,U}$

For $\phi_2$ arbitrary,

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New Problem:

Normalizing by $\frac{\hat{\phi}_2(0)}{\phi_2(0)}$, minimize

$$\int_{\xi_1 \in \mathbb{R}} \hat{\phi}_1(\xi_1) \widehat{W}(\xi_1) \frac{\phi_1(0)}{\phi_1(0)}$$

over $\phi_1$, where $\widehat{W}(\xi_1) = \delta(\xi_1) + \frac{\hat{\phi}_2(-\xi_1)}{\phi_2(0)} (|\xi_1| - 1) \mathbf{1}(\xi_1)$. 
Approach 1 Example: $W_{2,U}$

\[
\widetilde{W}(\xi_1) = \delta(\xi_1) + \frac{\hat{\phi}_2(-\xi_1)}{\hat{\phi}_2(0)}(|\xi_1| - 1)1(x)(\xi_1) = \delta(\xi_1) + m(\xi_1)
\]

- $\phi_2$ even $\implies$ $m$ is even.
Approach 1 Example: \( W_{2,U} \)

\[
\tilde{W}(\xi_1) = \delta(\xi_1) + \frac{\hat{\phi}_2(-\xi_1)}{\hat{\phi}_2(0)}(|\xi_1| - 1)1(x)(\xi_1) = \delta(\xi_1) + m(\xi_1)
\]

- \( \phi_2 \) even \( \implies \) \( m \) is even.
- 1-level case \( \implies \) optimal \( \phi_1 \) has \( \hat{\phi}_1(\xi_1) = (g * \tilde{g})(\xi_1) \)
  where \( g \in L^2[-\frac{1}{2}, \frac{1}{2}] \) satisfying

\[
1(x) = g(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} m(x - y)g(y) \, dy.
\]

Minimum value is \( \frac{1}{\langle 1, g \rangle_{L^2}} \).
**Approach 1 Example: \( W_{2,U} \)**

\[
1(x) = g(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} m(x - y)g(y) \, dy.
\]

Solution is found by iteration:
Approach 1 Example: $W_{2,U}$

$$1(x) = g(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} m(x - y)g(y) \, dy.$$ 

Solution is found by iteration:

- $K(x, y) := -m(x - y)$.  
- $K_n(x) := \int_{[-\frac{1}{2}, \frac{1}{2}]} K(x, t_1) \cdots K(t_{n-1}, t_n) \, dt_1 \cdots dt_n$.  


Approach 1 Example: $W_{2, U}$

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1(x) = g(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} m(x - y) g(y) \, dy.
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Solution is found by iteration:

- $K(x, y) := -m(x - y)$.
- $K_n(x) := \int_{[-\frac{1}{2}, \frac{1}{2}]^n} K(x, t_1) \ldots K(t_{n-1}, t_n) \, dt_1 \ldots dt_n$.
- $g(x) = 1(x) + \sum_{n=1}^{\infty} K_n(x)$.
- $\langle 1, g \rangle_{L^2} = 1 + \sum_{n=1}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} K_n(x) \, dx$. 
Approach 1 Example: $W_{2,U}$

\[ \langle 1, g \rangle_{L^2} = 1 + \sum_{n=1}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} K_n(x) \, dx \]

- Numerical data → $\hat{\phi}_2(\xi_2) = (h \ast \check{h})(\xi_2)$ where $h(y) = (1 - |y|)1(y)$ is a good choice.
Approach 1 Example: $W_{2,U}$

\[
\langle 1, g \rangle_{L^2} = 1 + \sum_{n=1}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} K_n(x) \, dx
\]

- Numerical data $\rightarrow \hat{\phi}_2(\xi_2) = (h * \check{h})(\xi_2)$ where $h(y) = (1 - |y|)1(y)$ is a good choice.
- Terms of series are nonnegative, so truncate after finitely many terms to get

\[
\frac{\hat{\phi}_2(0)}{\phi_2(0)} \frac{1}{\langle 1, g \rangle_{L^2}} \leq \frac{\hat{\phi}_2(0)}{\phi_2(0)} \left( 1 + \sum_{n=1}^{3} \int_{-\frac{1}{2}}^{\frac{1}{2}} K_n(x) \, dx \right)^{-1} \approx .519105
\]
### Numerical Data for $n = 2$ (Truncate at 3 terms)

<table>
<thead>
<tr>
<th></th>
<th>$h(y)$</th>
<th>Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_{2,0}$</td>
<td>$(1 -</td>
<td>y</td>
</tr>
<tr>
<td>$W_{2,SO(\text{Even})}$</td>
<td>$1(y)$</td>
<td>0.371402</td>
</tr>
<tr>
<td>$W_{2,SO(\text{Odd})}$</td>
<td>$1(y)$</td>
<td>0.447178</td>
</tr>
<tr>
<td>$W_{2,U}$</td>
<td>$(1 -</td>
<td>y</td>
</tr>
<tr>
<td>$W_{2,Sp}$</td>
<td>$1(y)$</td>
<td>0.447178</td>
</tr>
</tbody>
</table>

**Red** = numerical approximation up to small error.
Applications to Order of Vanishing

Assume $\mathcal{F}$ finite. $\Pr(N) :=$ probability that $L(s, f)$ has zero of order $N$ at $s = \frac{1}{2}$.
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$$\Pr(0) + \Pr(1) \geq \begin{cases} 
0.709299 & W_{2,0} \\
0.628598 & W_{2, SO(Even)} \\
0.552822 & W_{2, SO(Odd)} \\
0.480895 & W_{2, U} \\
0.552822 & W_{2, Sp}
\end{cases}$$
Applications to Order of Vanishing

Better results if every $f \in \mathcal{F}$ has same parity functional equation.
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Even:

$$\Pr(0) \geq \begin{cases} 
0.854650 & W_{2,O} \\
0.814299 & W_{2,SO(\text{Even})} \\
0.776411 & W_{2,SO(\text{Odd})} \\
0.740448 & W_{2,U} \\
0.776411 & W_{2,Sp} 
\end{cases}$$
Applications to Order of Vanishing

Better results if every $f \in F$ has same parity functional equation.

Even:

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0.740448 & W_{2,U} \\
0.776411 & W_{2,Sp}
\end{cases}$

Odd:

$\Pr(1) \geq \begin{cases} 
0.951550 & W_{2,O} \\
0.938100 & W_{2,SO(Even)} \\
0.925470 & W_{2,SO(Odd)} \\
0.913483 & W_{2,U} \\
0.925470 & W_{2,Sp}
\end{cases}$
References I


- Christopher P. Hughes and Steven J. Miller. Calculating the level density a la Katz-Sarnak.
References II
