Spies and Traitors: Random Matrix Kaleidoscopes and their Turncoat Eigenvalues

UMass REU Conf

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The Code of Matrices

Theorem (Berlekamp and van Tilborg, 1978)

Let R be an $N \times N$ random matrix, m^T an N-tuple, and $m' = m^T R$. Given solely m', recovering m^T is NP-hard.

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Applications to Cryptography

- Private key: G

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Applications to Cryptography

- Private key: G
- Public key: $(\prod S_i)\, GP$ *(S_i, P are random matrices)*
- Encryption: $m' = m^T (\prod S_i)\, GP$

Key Question

$$S_i = ???$$

Motivation to study iterated disco

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RLCE (Wang, 2007)

$$S_1 = \begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix}$$

Source Matrices

Component Matrices A, B_d

A is a random $N \times N$ Hermitian matrix, $\{B_d\}$'s are a family of $2^{d-1}N \times 2^{d-1}N$ random Hermitian matrices. Assume all B_d's have same limiting eigenvalue distribution, and that $\mathcal{M}_k(A) < \infty$, $\mathcal{M}_k(B_d) < \infty$. Note that we assume all entries are from mean 0, variance 1 distribution.

$$\mathcal{D}_1 = \begin{bmatrix} A & B_1 \\ B_1 & A \end{bmatrix}$$

$$\mathcal{D}_2 = \begin{bmatrix} A & B_1 & & \\ B_1 & A & & B_2 \\ & & A & B_1 \\ B_2 & & B_1 & A \end{bmatrix}$$

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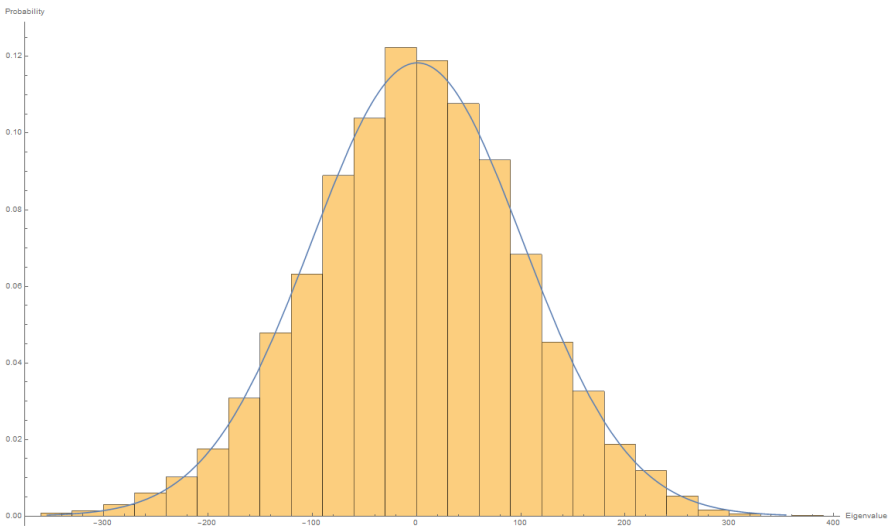
$$\mathcal{D}_3 = \begin{bmatrix} A & B_1 & B_2 & & B_3 \\ B_1 & A & & & \\ B_2 & & A & B_1 & \\ & & B_1 & A & \\ B_3 & & & & A & B_1 & B_2 \\ & & & & B_1 & A & \\ & & & & B_2 & & A & B_1 \\ & & & & & & B_1 & A \end{bmatrix}$$

Example

A: Gaussian (Massey, Miller, and Sinsheimer, 2007)

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$



10K × 10K SPT

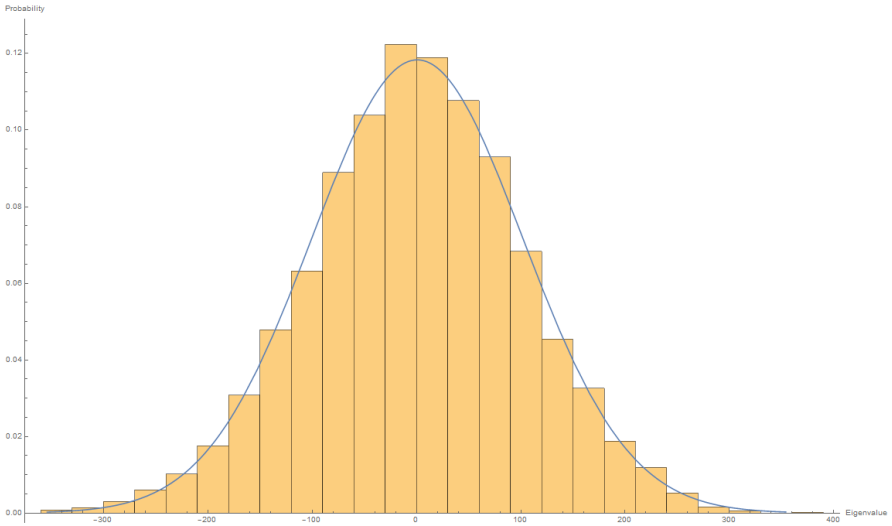
Example

A: Gaussian (Massey, Miller, and Sinsheimer, 2007)

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

B: Semi-circle (Wigner, 1955)

$$f(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4-x^2}, & |x| \leq 2 \\ 0, & |x| > 2. \end{cases}$$



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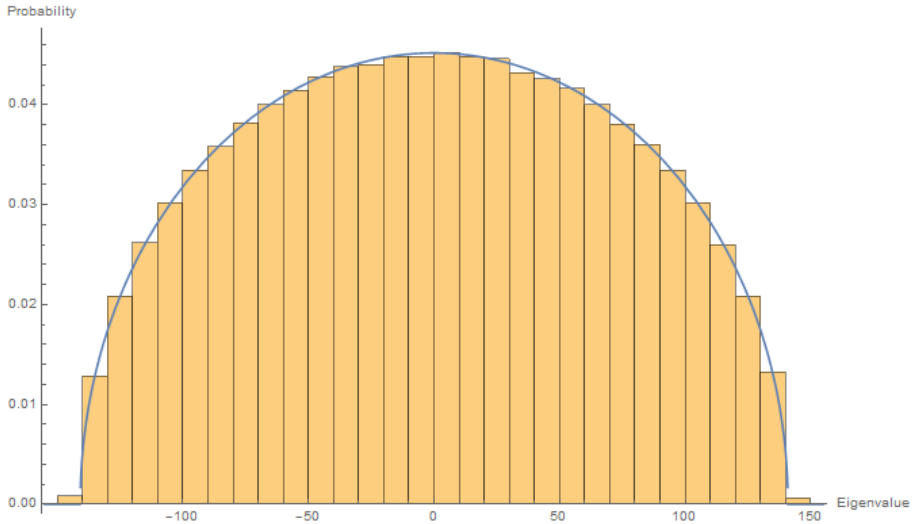
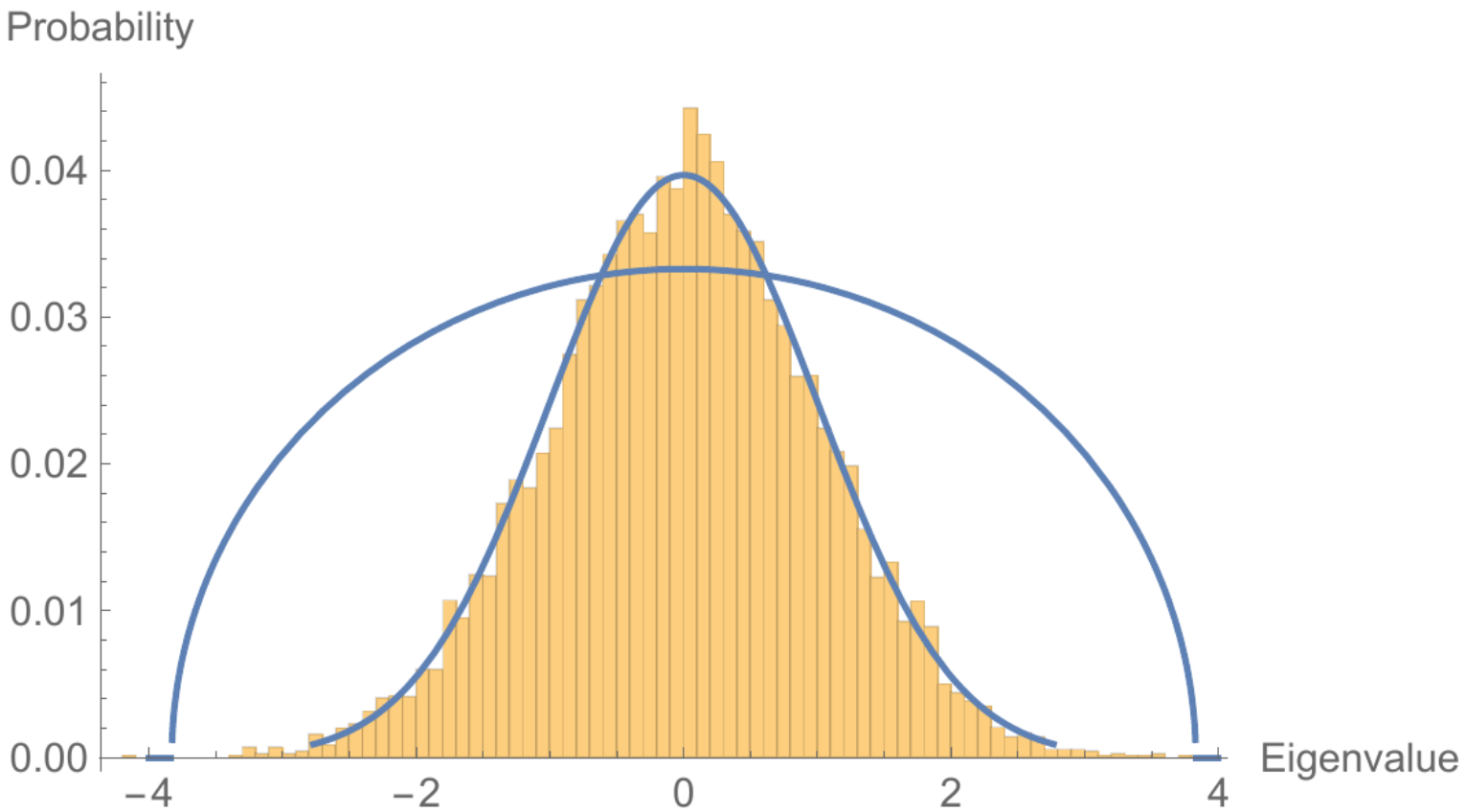


10K × 10K Real Symmetric

Infinite disco of STP and RS



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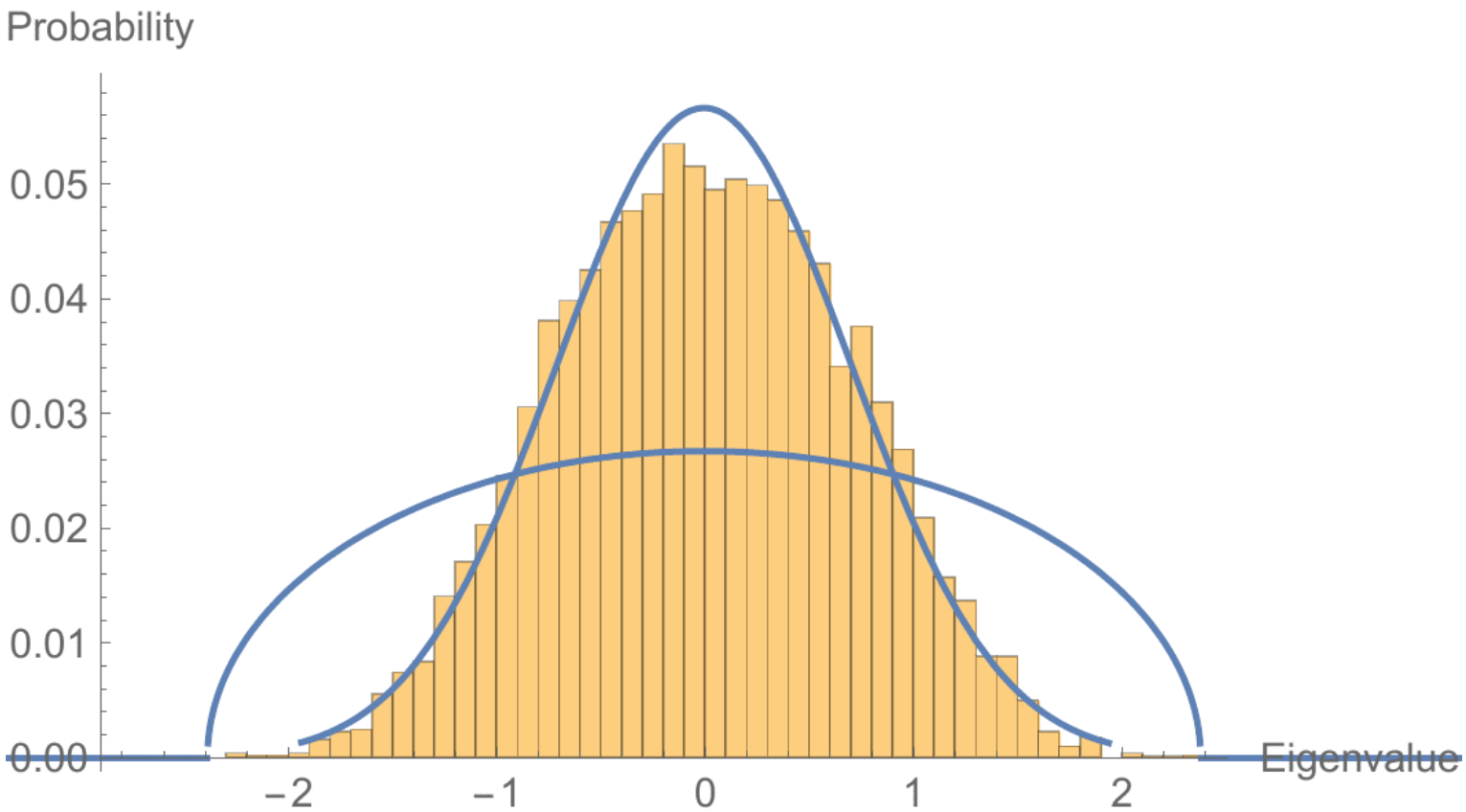

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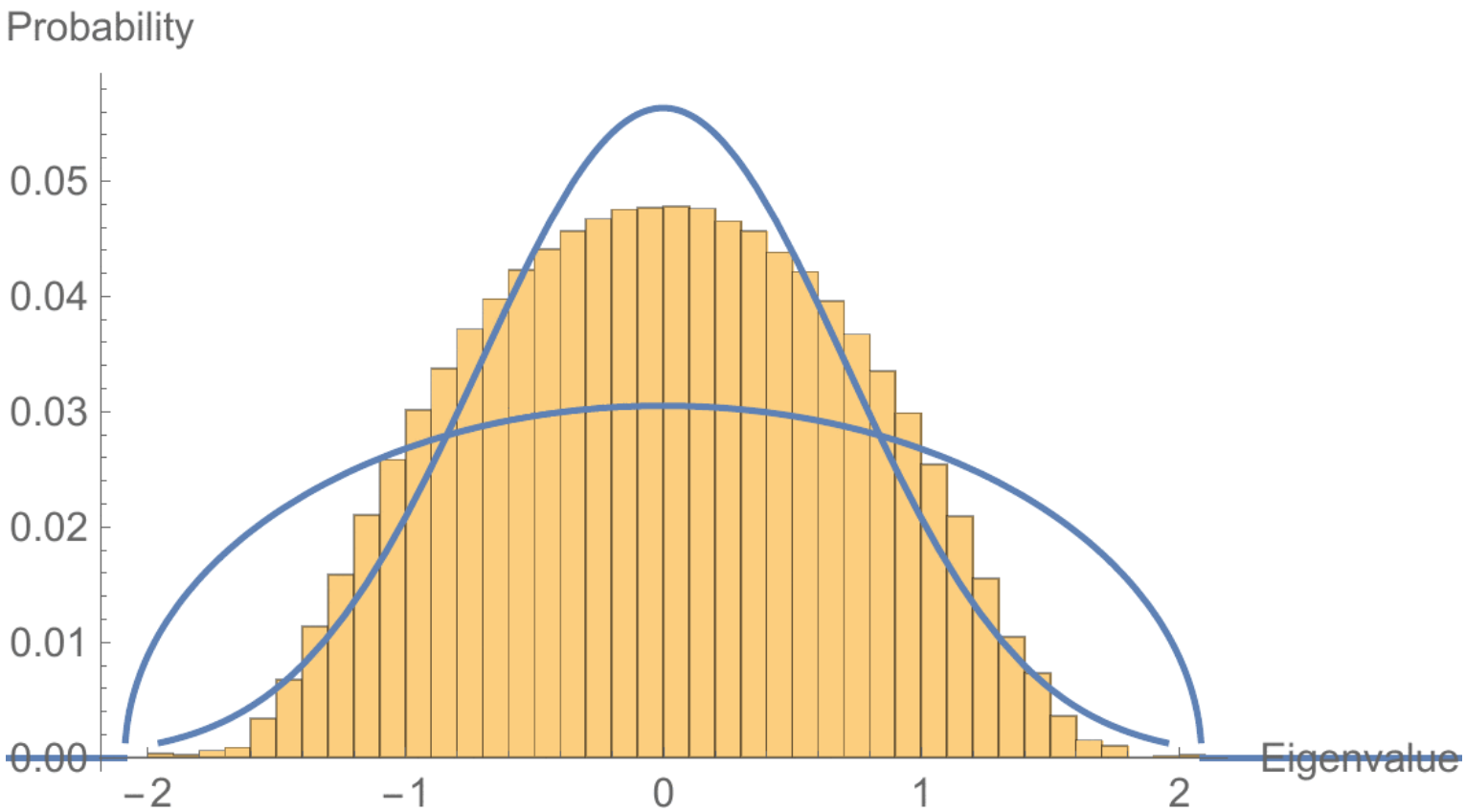

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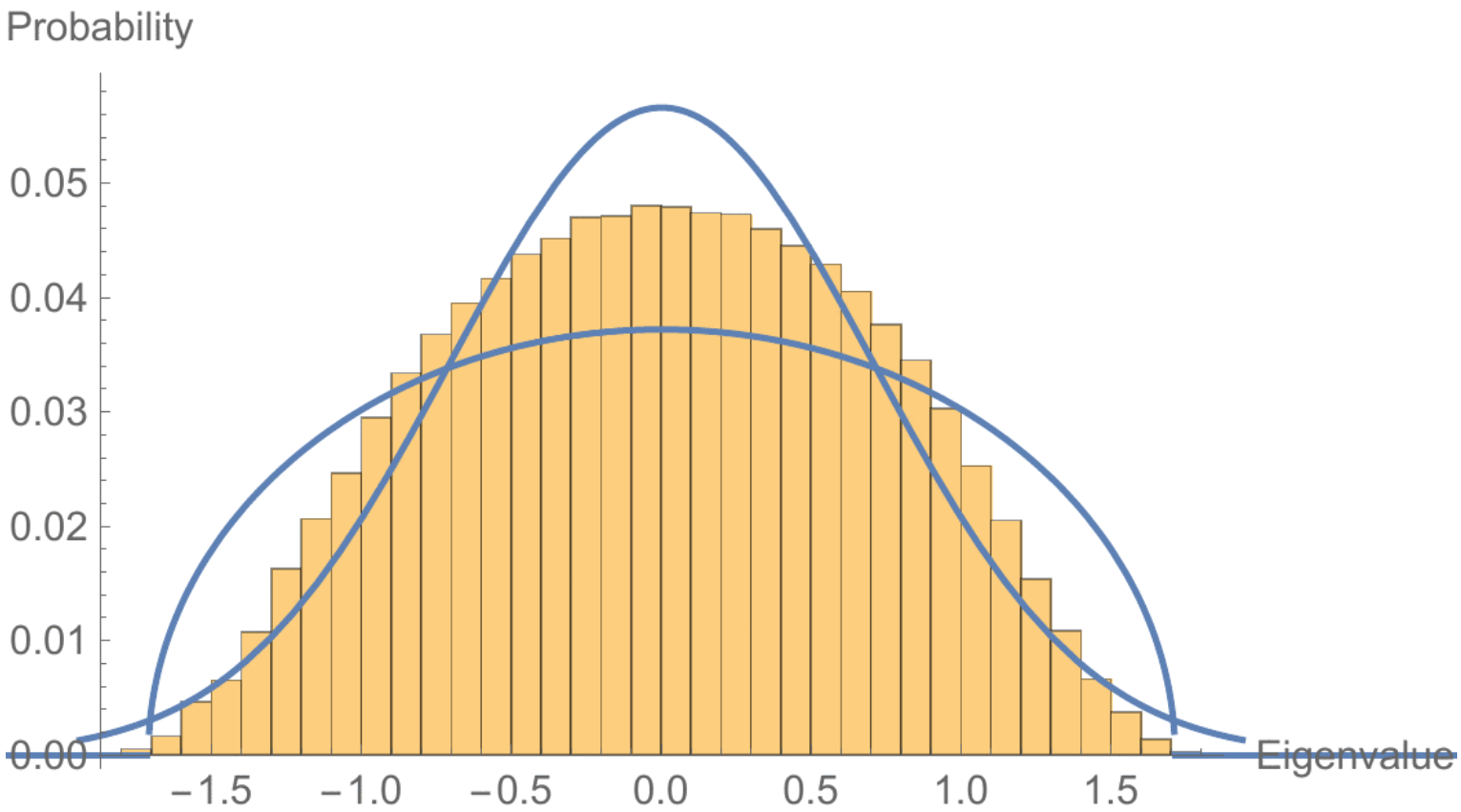

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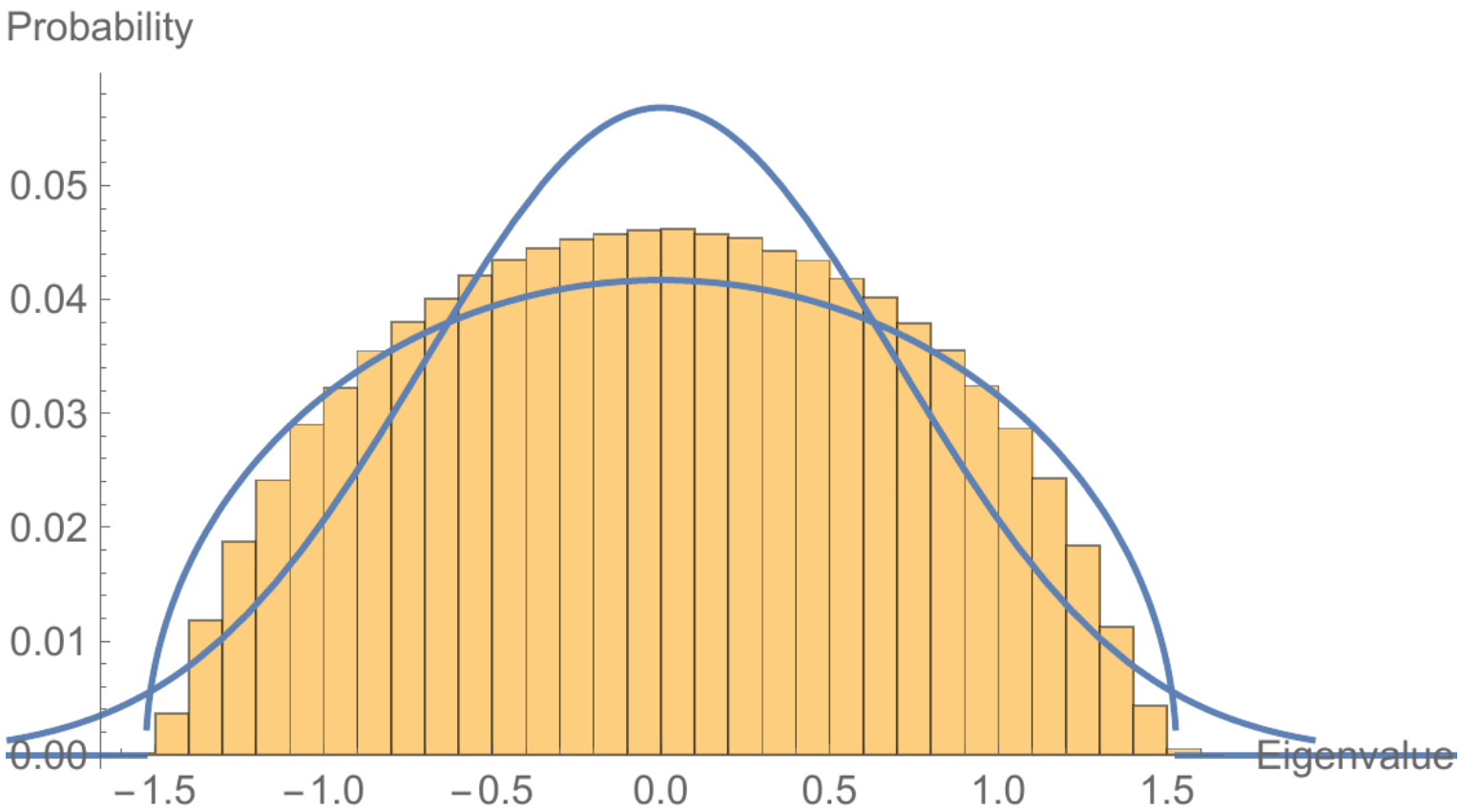

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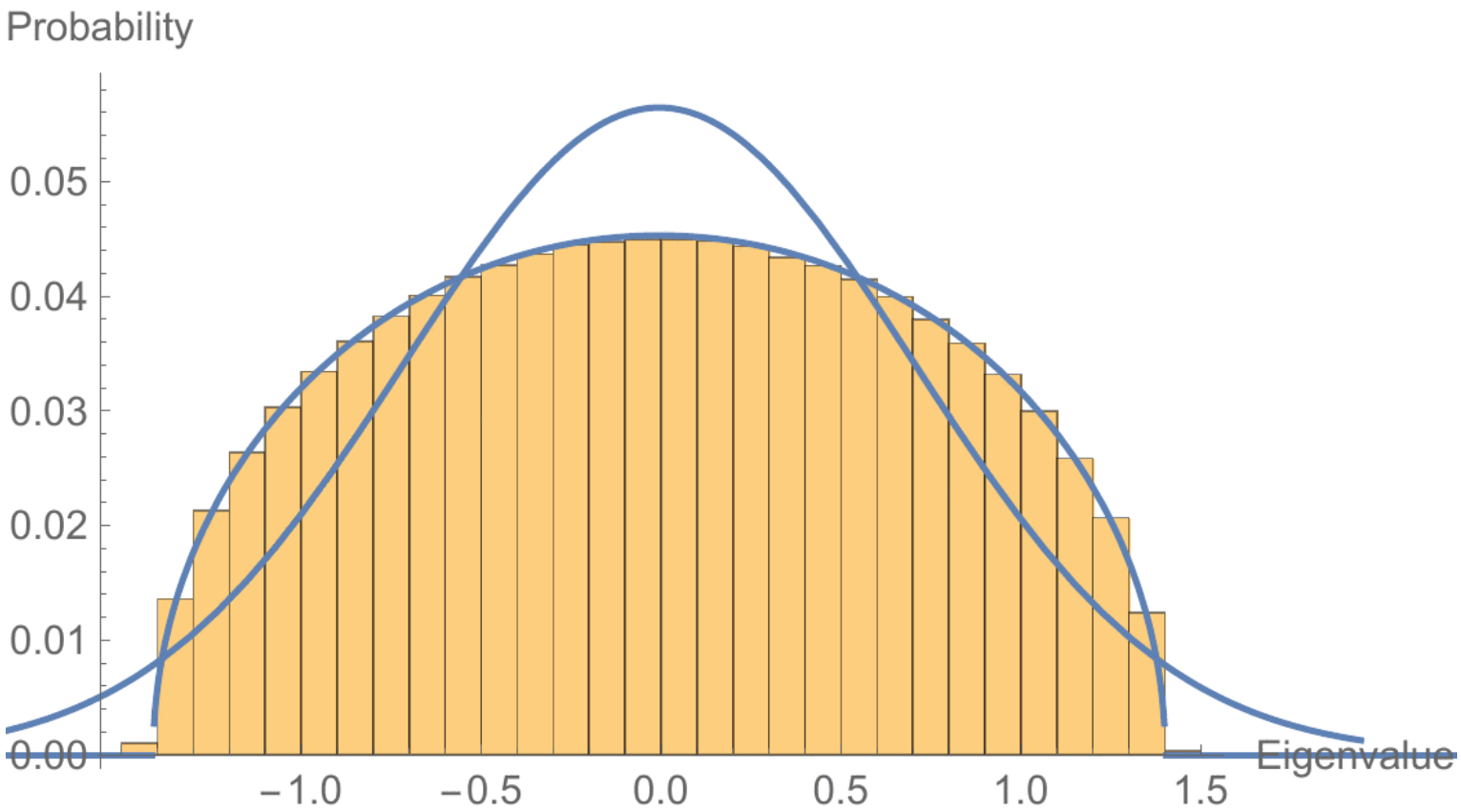

Limiting behavior of distribution of Inifnite Disco

Theorem(B., Borade, Devlin, Ma, Miller, Silva, and X., 2019)

The limiting eigenvalue distribution of $\mathcal{D}_d(A, \{B_d\})$ converges with order $\mathcal{O}\left(2^{-d}\right)$ to the limiting normalized eigenvalue distribution of the B_d ensemble as $N \to \infty$ and $d \to \infty$.

Defining Probability Space

We normalize the eigenvalues of $\mathcal{D}_d$ by $\sqrt{2^d N}$:

$$\mu_{\mathcal{D}_d, 2^d N}(x)dx = \frac{1}{2(2^d N)} \sum_{j=1}^{2^d N} \delta\left(x - \frac{\lambda_j(\mathcal{D}_d)}{\sqrt{2^d N}}\right)$$

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The kth Moment of D_i

$$M_k(\mathcal{D}_d) = \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{(2^d N)^{\frac{k}{2}+1}} \sum_{j=1}^{2^d N} \lambda_j^k(\mathcal{D}_d)\right]$$

What do we know about $\lambda_j(\mathcal{D}_d)$?

Eigenvalue Trace Lemma

$$\sum_{j=1}^{2^d N} \lambda_j^k(\mathcal{D}_d) \;=\; \text{Trace}(\mathcal{D}_d^k) \;=\; \sum_{1 \le j_1, \ldots, j_k \le 2^d N} d_{j_1,j_2} d_{j_2,j_3} \cdots d_{j_k,j_1}$$

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Odd k vanishes

$$M_k(\mathcal{D}_d) \;=\; \lim_{N\to\infty} \frac{1}{(2^d N)^{\frac{k}{2}+1}} \sum_{1\le j_1,\ldots,j_k\le 2^d N} \mathbb{E}\left[d_{j_1,j_2} d_{j_2,j_3} \cdots d_{j_k,j_1}\right] = 0.$$

Decomposing the $\mathcal{D}_d$ matrix

Let B_0 be from the same distribution of $\{B_d\}$. Define $C = A - B_0$ We can decompose $\mathcal{D}_d$ as follows:

$$\mathcal{D}_d = \mathcal{B} + \mathcal{C}$$

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Moments of $C = A - B_0$

C is Hermitian and all moments of C are finite.

Defining $\mathcal{B}$

Choose from the ensemble of B_d's a random $N \times N$ matrix B_0 and define

$$\mathcal{B}_3 = \begin{bmatrix} B_0 & B_1 & B_2 & & & \\ B_1 & B_0 & & & & \\ & B_2 & B_0 & B_1 & & \\ & & B_1 & B_0 & & \\ & & & & B_3 & \\ & & & & & B_0 & B_1 & B_2 \\ & & & & & B_1 & B_0 & \\ & & & & & B_2 & & B_0 & B_1 \\ & & & & & & & B_1 & B_0 \end{bmatrix}$$

Sketch of proof

Consider the decomposition:

$$\mathcal{M}_k(\mathcal{D}) = \lim_{d,\, N\to\infty} \frac{1}{(2^d N)^{\frac{k}{2}+1}} \mathbb{E}\left[\mathrm{Tr}\left((\mathcal{B}+\mathcal{C})^k\right)\right]$$

$$= \lim_{d,\, N\to\infty} \frac{\mathbb{E}\left[\mathrm{Tr}(\mathcal{B}^k)\right]}{(2^d N)^{\frac{k}{2}+1}} + \frac{\mathbb{E}\left[\mathrm{Tr}(\mathit{MixedProducts})\right]}{(2^d N)^{\frac{k}{2}+1}} + \frac{\mathbb{E}\left[\mathrm{Tr}(\mathcal{C}^k)\right]}{(2^d N)^{\frac{k}{2}+1}}.$$

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$$\mathcal{M}_k(B_i) \quad \leftrightarrow \quad \text{Contribution of } \frac{\mathbb{E}\left[\mathrm{Tr}(\mathcal{B}^k)\right]}{(2^d N)^{\frac{k}{2}+1}}$$

$$\frac{\mathcal{M}_k(C)}{2^{\frac{dk}{2}}} \to 0 \quad \leftrightarrow \quad \text{Contribution of } \frac{\mathbb{E}\left[\mathrm{Tr}(\mathcal{C}^k)\right]}{(2^d N)^{\frac{k}{2}+1}}$$

$$\frac{\mathit{FiniteConstant}}{2^d} \to 0 \quad \leftrightarrow \quad \text{Contribution of } \frac{\mathbb{E}\left[\mathrm{Tr}(\mathit{MixedProducts})\right]}{(2^d N)^{\frac{k}{2}+1}}$$

Contribution of $\frac{\mathbb{E}\left[\mathrm{Tr}(\mathcal{B}^k)\right]}{(2^d N)^{\frac{k}{2}+1}}$ is $\mathcal{M}_k(B_i)$ as $N,\ d \to \infty$

Theorem(Luntzlara, Blackwell, B., Devlin, M., Miller, Silva, Wang and Xu, 2019)

The normalized eigenvalue distribution of $\mathcal{D}_1(A, B)$ is the same as that of A and B if A and B are chosen from ensembles with the same normalized eigenvalue distributions.

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$$\mathcal{B} = \begin{bmatrix} B_0 & B_1 & & & & & B_3 \\ B_1 & B_0 & B_2 & & & & \\ & B_2 & & B_0 & B_1 & & \\ & & & B_1 & B_0 & & \\ & & & & B_3 & & \\ B_0 & B_1 & & & & B_2 & \\ B_1 & B_0 & & & & & \\ & & B_2 & & & B_0 & B_1 \\ & & & & & B_1 & B_0 \end{bmatrix}$$

Contribution of $\frac{\mathbb{E}\left[\mathrm{Tr}(\mathcal{C}^k)\right]}{(2^d N)^{\frac{k}{2}+1}}$ is $\frac{\mathcal{M}_k(C)}{2^{\frac{dk}{2}}} \to 0$ as $N\ d \to \infty$

$$\mathcal{C}^k = \begin{bmatrix} C^k & & \\ & \ddots & \\ & & C^k \end{bmatrix} \implies \mathrm{Tr}(\mathcal{C}^k) = 2^d \mathrm{Tr}(C^k)$$

$$\implies \frac{\mathbb{E}\left[\mathrm{Tr}(\mathcal{C}^k)\right]}{(2^d N)^{\frac{k}{2}+1}} = \frac{2^d \mathcal{M}_k(C)}{2^{\frac{dk}{2}+d}}$$

$$= \frac{\mathcal{M}_k(C)}{2^{\frac{dk}{2}}} \to 0 \text{ as } d \to \infty.$$

Challenges

Expansion of the product $(\mathcal{B}+\mathcal{C})^k$ yields a non-commutative, bi-variate matrix polynomial:

$$\mathbb{E}\left[\mathrm{Tr}(\mathit{MixedProducts})\right] = \sum \mathbb{E}\left[\mathrm{Tr}\left(\mathcal{B}^{I_1}\mathcal{C}^{J_1}\cdots\mathcal{B}^{I_p}\mathcal{C}^{J_p}\right)\right]$$

Where $\sum I_p = I$, $\sum J_p = J$, and $I + J = k$.

Bound on mixed terms

Theorem.(Blackwell, B., Devlin, M., Miller, Silva, and Xu, 2019)

$A,\ B$ be $m \times m$ Hermitian random matrices. $I_i,\ J_i > 0$ s.t. $\sum_{i=1}^{p} I_i = I$ and $\sum_{i=i}^{p} J_i = J$, where$I + J = K$ $I, J,$ and K are all even. Then

$$\mathbb{E}[\mathrm{Tr}(A^{I_1} B^{J_1} \cdots A^{I_p} B^{J_p})] \leq \mathbb{E}[\mathrm{Tr}(A^K)]^{\frac{I}{K}} \mathbb{E}[\mathrm{Tr}(B^K)]^{\frac{J}{K}}.$$

Tools used to prove bound

Definition:p-Schatten norm of an operator X

$$||X||_p = \left(\sum_{i=1}^{m} \sigma_i^p(X) \right)^{\frac{1}{p}}$$

Here $\sigma_i(X)$ are singular values of X which are square roots of the eigenvalues of the matrix $\overline{X^T X}$.

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Generalized Hölder-trace Inequality

Let $X_1, X_2, ..., X_k$ be a set of $m \times m$ matrices. Let $p_1, p_2, ..., p_k \geq 0$ such that $\sum_{i=1}^{k} \frac{1}{p_i} = 1$.

$$|\mathrm{Tr}(X_1 X_2 \cdots X_k)| \leq \prod_{i=1}^{k} ||X_i||_{p_i}.$$

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Note that k is even. And for Hermitian A, $\sigma_i(A) = |\lambda_i(A)|$.

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$$\begin{aligned}
\mathbb{E}[\mathrm{Tr}(A^{I_1}B^{J_1}\cdots A^{I_p}B^{J_p})] &\leq \mathbb{E}\left[||A||_k^I||B||_k^J\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^m \sigma_i^k(A)\right)^{\frac{I}{k}}\right]\mathbb{E}\left[\left(\sum_{i=1}^m \sigma_i^k(B)\right)^{\frac{J}{k}}\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^m \lambda_i^k(A)\right)^{\frac{I}{k}}\right]\mathbb{E}\left[\left(\sum_{i=1}^m \lambda_i^k(B)\right)^{\frac{J}{k}}\right] \\
&= \mathbb{E}\left[\mathrm{Tr}(A^k)^{\frac{I}{k}}\right]\mathbb{E}\left[\mathrm{Tr}(B^k)^{\frac{J}{k}}\right] \\
&\leq \mathbb{E}\left[\mathrm{Tr}(A^k)\right]^{\frac{I}{k}}\mathbb{E}\left[\mathrm{Tr}(B^k)\right]^{\frac{J}{k}} \quad \text{(Jensen's inequality).}
\end{aligned}$$

Contribution of $\frac{\mathbb{E}[\mathrm{Tr}(\mathit{MixedProducts})]}{(2^d N)^{\frac{k}{2}+1}}$ is $\frac{\mathit{FiniteConstant}}{2^d} \to 0$

$$\frac{\mathbb{E}[\mathrm{Tr}(\mathcal{B}^{l_1}\mathcal{C}^{J_1}\cdots\mathcal{B}^{l_p}\mathcal{C}^{J_p})]}{(2^d N)^{\frac{k}{2}+1}} \leq \left(\frac{\mathbb{E}[\mathrm{Tr}(\mathcal{B}^K)]}{(2^d N)^{\frac{k}{2}+1}}\right)^{\frac{I}{K}} \left(\frac{\mathbb{E}\left[\mathrm{Tr}(C^k)\right]}{N^{\frac{k}{2}+1}}\right)^{\frac{J}{k}} \frac{1}{2^{dI}}$$

$$= \frac{\mathcal{M}_k(B_i)^{\frac{I}{k}}\mathcal{M}_k(C)^{\frac{J}{k}}}{2^{dI}} \to 0 \text{ as } \mathrm{d} \to \infty.$$

$$\implies \frac{\mathbb{E}\left[\mathrm{Tr}(\mathit{MixedProducts})\right]}{(2^d N)^{\frac{k}{2}+1}} \to 0 \text{ as } N, \quad d \to \infty.$$

Result

$$\mathcal{M}_k(\mathcal{D}) = \lim_{d,\, N\to\infty} \frac{\mathbb{E}\left[\mathrm{Tr}(\mathcal{B}^k)\right]}{(2^d N)^{\frac{k}{2}+1}} + \frac{\mathbb{E}\left[\mathrm{Tr}(\mathit{MixedProducts})\right]}{(2^d N)^{\frac{k}{2}+1}} + \frac{\mathbb{E}\left[\mathrm{Tr}(\mathcal{C}^k)\right]}{(2^d N)^{\frac{k}{2}+1}}$$

$$= \lim_{d\to\infty} \mathcal{M}_k(B_i) + \mathcal{O}(\frac{1}{2^d})$$

$$= \mathcal{M}_k(B_i)$$

Thank you!

Thanks to Williams College, UIC and Bowdoin College for funding

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