**Introduction**

Considering a set $A \subseteq \{0,1, \ldots, n-1\}$, we define a **subset** as $A + A = \{i + j \mid i,j \in A\}$.

We look at random sets $A$ where $\mathbb{P}(i \in A) = p$.

**Notation:** \( q = 1 - p \), $M_{[0,n-1]} = [0, 2n - 2]\) \((A + A)\) = 2n - 1 - \(|A + A|\), $m_n(k) = \mathbb{P}(M_{[0,n-1]} = k)$.

**Uniform Distribution:** Most previous work focused on $p = 1/2$. This implies every subset $A$ has an equal probability of occurring.

**Expected Value Generalization**

We generalize previous results for any $p$. The explicit formula for $\mathbb{E}[|A + A|]$ is stated below.

**Expected Value Formula**

For $p \in [0,1]$, $\mathbb{E}[|A + A|]$ is

\[
\sum_{r=0}^{n} p^r (1 - p)^{n-r} \binom{n}{r} \left( \frac{1}{n-r} \right) - \frac{f(n-1)}{n-r}.
\]

where $f(k) = \sum_{r=0}^{n} \binom{n}{r} (1 - p)^{n-r} \sum_{i=r}^{n} \frac{(n-i)!}{i! (n-i-r)!} \frac{1}{i-r}$ for $k$ odd and $\sum_{r=0}^{n} \binom{n}{r} (1 - p)^{n-r} \sum_{i=r}^{n} \frac{(n-i)!}{i! (n-i-r)!} \frac{1}{i}$ for $k$ even.

We also find bounds for the Expected Value.

**Approach**

Previously, every subset had an equal chance of occurring, which allowed for a simpler calculation. Now, we must take into account the probability that $|A| = r$, where $r \in [0,n]$. Using similar techniques, we found $\mathbb{E}[|A + A|] = \sum_{r=0}^{n} p^r (1 - p)^{n-r} \binom{n}{r} \left( \frac{1}{n-r} \right) - \frac{f(n-1)}{n-r}$.

We compute $\mathbb{P}(k \not\in A + A \mid |A| = r)$ using graph theory. We define $G = (V,E)$ with vertices representing elements in $[0,n-1]$ and an edge is present if $v_1 + v_2 = k$. We are interested in finding a vertex cover with $n - r$ vertices.

\[
f(k) = \left\{ \begin{array}{cl}
\sum_{r=0}^{n} \binom{n}{r} (1 - p)^{n-r} \sum_{i=r}^{n} \frac{(n-i)!}{i! (n-i-r)!} \frac{1}{i-r} & \text{for } k \text{ odd} \\
\sum_{r=0}^{n} \binom{n}{r} (1 - p)^{n-r} \sum_{i=r}^{n} \frac{(n-i)!}{i! (n-i-r)!} \frac{1}{i} & \text{for } k \text{ even}
\end{array} \right.
\]

From this, we can derive $f(k)$ from the Expected Value formula.

**mₙ(k) Generalization**

We generalize previous results of $m_n(k)$ for any $p$, and show a plot for $k = 10$.

**mₙ(k) Bounds**

Let $n > \frac{2 \phi(n)}{\mathbb{E}[|A + A|]}$, and $\phi(p) := \sqrt{1 + 2p - 3p^2}$.

Then, $q^{k/2} \ll m_n(k) \ll \frac{(1 - p + \phi(p))^k}{2}$.

After some manipulation, we find

$$m_n(k) \leq \mathbb{P}(A + A \text{ misses } 2 \text{ elements greater than } k - 3) = \sum_{k-3 \leq j \leq k} \mathbb{P}(i,j \notin A + A) \ll \frac{(1 - p + \phi(p))^k}{2}.$$  

Where the last term comes from a generalization of graph theory introduced by Lazarev, Miller and O’Bryant. We seek to find $\mathbb{P}(i,j \notin A + A)$ and using a similar definition of a graph as earlier, this is the same as finding a vertex cover of missing elements on a path, as below.

\[
 v_1 v_2 v_3 v_4 v_n
\]

Letting $n$ denote the number of paths and $a_n$ denoting the probability of a vertex cover, we derive $a_n = qa_{n-1} + pq a_{n-2}$, and get a closed form

$$a_n = \left( (\phi(p) - 1 - p) (1 - p - \phi(p))^n + \frac{2n+1}{2n+1} \phi(p) \right).$$

**References**


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