The Fibonacci Quilt Game

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Outline

The Fibonacci Sequence

$$1; 1; 2; 3; 5; 8; 13; 21; 34; 55:::$$

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$$F_n = F_{n\ \ 1} + F_{n\ \ 2}$$

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$$F_n = F_{n\ \ 1} + F_{n\ \ 2}$$

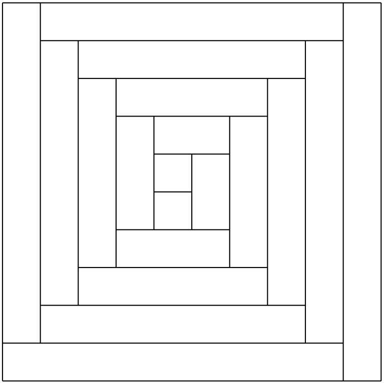
Theorem (Zeckendorf)

Every positive integer can be written uniquely as the sum of non-consecutive Fibonacci numbers where

$$F_n = F_{n\ \ 1} + F_{n\ \ 2}$$

and $F_1 = 1$, $F_2 = 2$.

The Fibonacci Quilt Sequence



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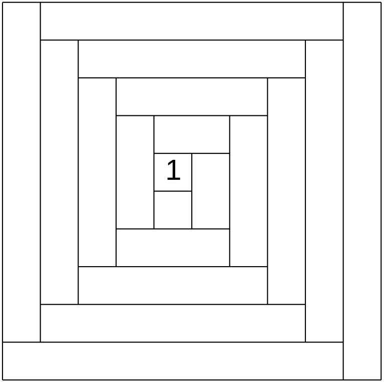

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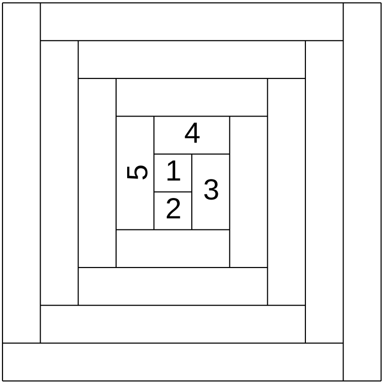

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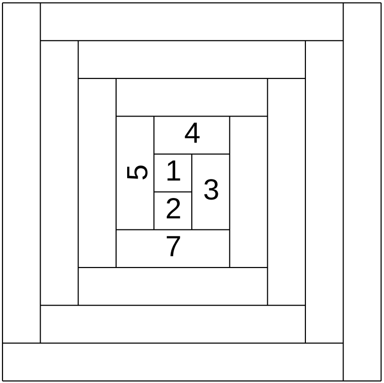

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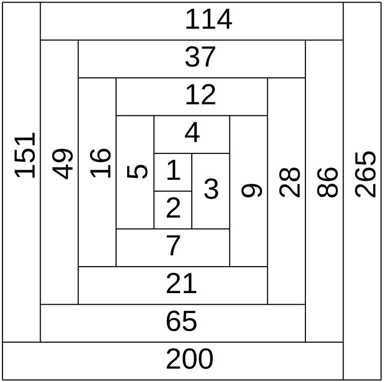

FQ-legal Decomposition

Definition (Catral, Ford, Harris, Miller, Nelson)

Let an increasing sequence of positive integers $\{q_i\}_{i=1}^{\infty}$ be given. We declare a decomposition of an integer

$$m = q_{l_1} + q_{l_2} + \cdots + q_{l_t}$$

(where $q_{l_i} > q_{l_{i+1}}$) to be an FQ-legal decomposition if for all i, j, $|l_i - l_j| \notin \{0, 1, 3, 4\}$ and $\{1, 3\} \not\subseteq \{l_1, l_2, \ldots, l_t\}$.

The Fibonacci Quilt Sequence

Definition (Catral, Ford, Harris, Miller, Nelson)

The Fibonacci Quilt sequence is an increasing sequence of positive integers $\{q_i\}_{i=1}^{\infty}$, where every q_i $(i \geq 1)$ is the smallest positive integer that does not have an FQ-legal decomposition using the elements $\{q_1, \dots, q_{i-1}\}$.

Recurrence Relations

Theorem (Catral, Ford, Harris, Miller, Nelson)

Let q_n denote the n^{th} term in the Fibonacci Quilt, then

$$\text{for } n \geq 5;\ q_{n+1} = q_{n-1} + q_{n-2};$$

$$\text{for } n \geq 6;\ q_{n+1} = q_n + q_{n-4}.$$

General Rules

Inspired by the Two Player Zeckendorf Game

Two players alternate turns, the last person to move wins

Start the game with n 1's (q_1's)

A turn consists of one of 5 rules, which preserve that $\sum q_i = n$ by exchanging a pair q_i, q_j such that i, j are an illegal distance apart for a single term or legal pair.

Rule 1

For $n \geq 2$; $q_n + q_{n+1} \rightarrow q_{n+3}$

General Rules

Rule 2

$$\text{For } n \geq 2;\ q_n + q_{n+4} \rightarrow q_{n+5}$$

General Rules

Rule 3

$$\text{For } n \quad 7;\ 2q_n\ !\ \ q_{n+2} + q_{n\ \ 5}$$

General Rules

Rule 4

$$\text{For } n \geq 7;\ q_n + q_{n+3} \to q_{n-5} + q_{n+4}$$

General Rules

Rule 5

$$q_1 + q_3 ! \quad q_4$$

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To handle base cases, we added additional base rules that

- preserves $^{\mathrm{P}}\; q_i = n$
- does not produce violation of legality

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Special Rule

$$1 + 5 \;!\; 2 + 4$$

Note: This rule can only be applied when nothing else can be done.

Example Game

1, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28...

n = 10 = 9 + 1

1	2	3	4	5	7	9
10	0	0	0	0	0	0

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Rule 3: q_1^2 ! q_2

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8	1	0	0	0	0	0	Rule 3: q_1^2 ! q_2
7	0	1	0	0	0	0	Rule 1: $q_1 + q_2$! q_3

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5	1	1	0	0	0	0	Rule 3: q_1^2 ! q_2
4	0	2	0	0	0	0	Rule 1: $q_1 + q_2$! q_3

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5	1	1	0	0	0	0	Rule 3: q_1^2 ! q_2
4	0	2	0	0	0	0	Rule 1: $q_1 + q_2$! q_3
3	0	1	1	0	0	0	Rule 5: $q_1 + q_3$! q_4

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4	0	2	0	0	0	0	Rule 1: $q_1 + q_2$! q_3
3	0	1	1	0	0	0	Rule 5: $q_1 + q_3$! q_4
2	0	1	0	1	0	0	Rule 2: $q_1 + q_4$! q_5

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0	1	1	0	1	0	0	Rule 3: q_1^2 ! q_2
0	0	1	0	0	1	0	Rule 4: $q_2 + q_5$! q_6

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2	0	1	0	1	0	0	Rule 2: $q_1 + q_4$! q_5
0	1	1	0	1	0	0	Rule 3: q_1^2 ! q_2
0	0	1	0	0	1	0	Rule 4: $q_2 + q_5$! q_6
1	0	0	0	0	0	1	Rule 4: $q_3 + q_6$! $q_1 + q_7$

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Theorem

Every game terminates in a finite number of moves at an FQ-legal decomposition.

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$$q_n \wedge q_{n+3} \to q_{n+4} \wedge q_{n-5}: \sqrt{n+4} + \sqrt{n-5} - \sqrt{n} - \sqrt{n+3} < 0$$

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There is more than one possible game for any $n > 3$.

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Conjecture

The game is fair.

Lower Bound on Game Length

Notation

Let $L(n)$ denote the maximum number of terms in an FQ-legal decomposition of n. Let $l(n)$ denote the minimum number of terms in an FQ-legal decomposition of n.

Examples:

$20 = 16 + 4 = 12 + 7 + 1$

$L(20) = 3$; $l(20) = 2$

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Let L(n) denote the maximum number of terms in an FQ-legal decomposition of n. Let l(n) denote the minimum number of terms in an FQ-legal decomposition of n.

Examples:
20 = 16 + 4 = 12 + 7 + 1
L(20) = 3; l(20) = 2
50 = 49 + 1 = 28 + 16 + 4 + 2
L(50) = 4, l(50) = 2

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If n is in the Fibonacci Quilt Sequence, denoted q_i

$$q_{i\quad 3} + q_{i\quad 2} = q_i$$

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The shortest possible game on n is completed in n L(n) moves.

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If n is in the Fibonacci Quilt Sequence, denoted q_i

$$q_{i} \;\; 3 + q_{i} \;\; 2 = q_i$$

If n is not in the sequence

$$n = q_{i_1} + q_{i_2} + \quad + q_{i_{L(n)}}$$

Number of moves:

$$(q_{i_1} \quad 1) + (q_{i_2} \quad 1) + \quad + (q_{i_{L(n)}} \quad 1)$$

$$= (q_{i_1} + q_{i_2} + \quad + q_{i_{L(n)}}) \quad L(n)$$

$$=$$

Distribution of Game Lengths

Conjecture

The distribution of a random game length approaches Gaussian as n increases.

Figure: Distribution of 1000 games on n=60

Future Work

Is there a deterministic game that always results in the lower bound?

What patterns emerge from the winner of certain deterministic games as n increases?

Does either player have a winning strategy?

- Analogous result on the Zeckendorf Game shows that for $n > 2$, player 2 has a winning strategy

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