The Fibonacci Quilt Game

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Mentored by Steven J. Miller, Williams College, at the SMALL REU 2019.

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Outline

The Fibonacci Sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55...$$

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Let $F_0 = F_1 = 1$, and for $n >= 2$

$$F_n = F_{n-1} + F_{n-2}$$

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Let $F_0 = F_1 = 1$, and for $n >= 2$

$$F_n = F_{n-1} + F_{n-2}$$

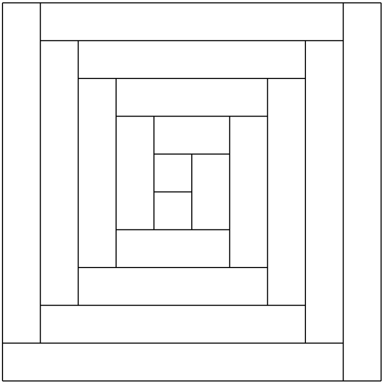
Theorem (Zeckendorf)

Every positive integer can be written uniquely as the sum of non-consecutive Fibonacci numbers where

$$F_n = F_{n-1} + F_{n-2}$$

and $F_1 = 1$, $F_2 = 2$.

The Fibonacci Quilt Sequence



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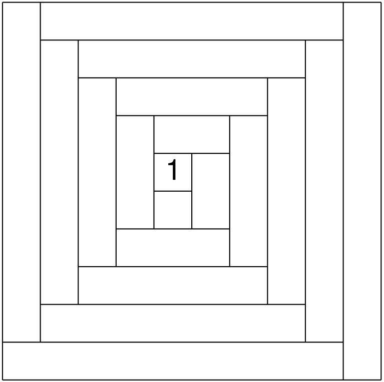

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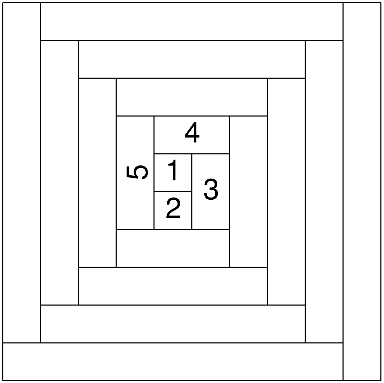

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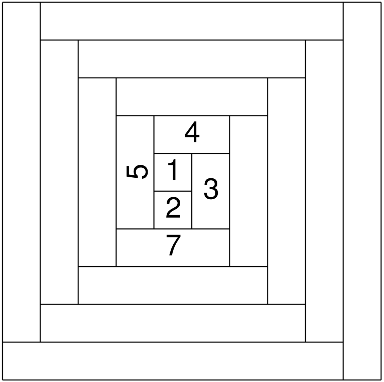

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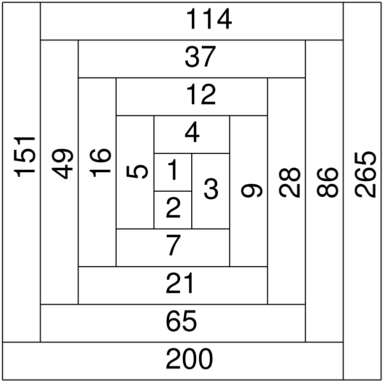


FQ-legal Decomposition

Definition (Catral, Ford, Harris, Miller, Nelson)

Let an increasing sequence of positive integers $q_i{}_{i=1}^{\infty}$ be given. We declare a decomposition of an integer

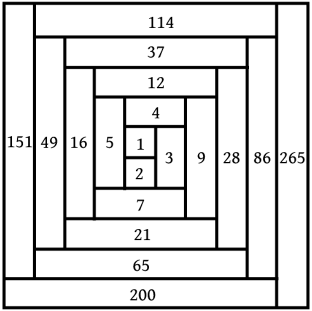
$$m \;=\; q_{l_1} + q_{l_2} + \cdots + q_{l_t}$$

(where $q_{l_i} > q_{l_{i+1}}$) to be an FQ-legal decomposition if for all i, j, $|l_i - l_j| \neq 0, 1, 3, 4$ and $\{1, 3\} \not\subset \{l_1, l_2, \ldots, l_t\}$.

The Fibonacci Quilt Sequence

Definition (Catral, Ford, Harris, Miller, Nelson)

The Fibonacci Quilt sequence is an increasing sequence of positive integers $\{q_i\}_{i=1}^{\infty}$, where every q_i $(i \geq 1)$ is the smallest positive integer that does not have an FQ-legal decomposition using the elements $\{q_1, \ldots, q_{i-1}\}$.



Recurrence Relations

Theorem (Catral, Ford, Harris, Miller, Nelson)

Let q_n denote the n^{th} term in the Fibonacci Quilt, then

$$\text{for } n \geq 5, q_{n+1} = q_{n-1} + q_{n-2},$$

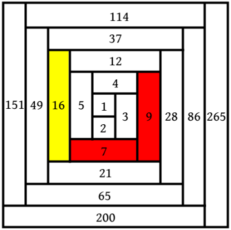
$$\text{for } n \geq 6, q_{n+1} = q_n + q_{n-4}.$$

General Rules

- Inspired by the Two Player Zeckendorf Game
- Two players alternate turns, the last person to move wins
- Start the game with n 1's (q_1's)
- A turn consists of one of 5 rules, which preserve that $\sum q_i = n$ by exchanging a pair q_i, q_j such that i, j are an illegal distance apart for a single term or legal pair.

Rule 1

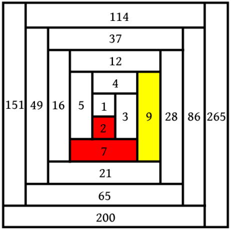
For $n \geq 2$, $q_n + q_{n+1} \rightarrow q_{n+3}$



General Rules

Rule 2

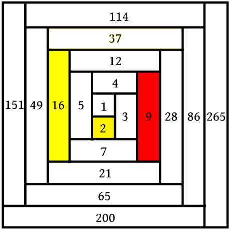
For $n \geq 2$, $q_n + q_{n+4} \rightarrow q_{n+5}$



General Rules

Rule 3

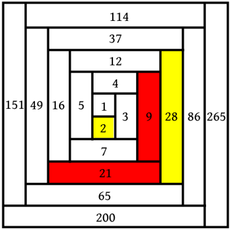
For $n \geq 7,\ 2q_n \rightarrow q_{n+2} + q_{n-5}$



General Rules

Rule 4

For $n \geq 7, \; q_n + q_{n+3} \rightarrow q_{n-5} + q_{n+4}$



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Special Rule

$$1 + 5 \ \to \ 2 + 4$$

Note: This rule can only be applied when nothing else can be done.

Example Game

1, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28...

n = 10 = 9 + 1

1	2	3	4	5	7	9
10	0	0	0	0	0	0

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4	0	2	0	0	0	0	Rule 1: $q_1 + q_2 \to q_3$

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4	0	2	0	0	0	0	Rule 1: $q_1 + q_2 \to q_3$
3	0	1	1	0	0	0	Rule 5: $q_1 + q_3 \to q_4$

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3	0	1	1	0	0	0	Rule 5: $q_1 + q_3 \to q_4$
2	0	1	0	1	0	0	Rule 2: $q_1 + q_4 \to q_5$

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0	1	1	0	1	0	0	Rule 3: $q_1^2 \rightarrow q_2$
0	0	1	0	0	1	0	Rule 4: $q_2 + q_5 \rightarrow q_6$
1	0	0	0	0	0	1	Rule 4: $q_3 + q_6 \rightarrow q_1 + q_7$

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Conjecture

The game is fair.

Lower Bound on Game Length

Notation

Let $L(n)$ denote the maximum number of terms in an FQ-legal decomposition of n. Let $l(n)$ denote the minimum number of terms in an FQ-legal decomposition of n.

Examples:
20 = 16 + 4 = 12 + 7 + 1
$L(20) = 3$, $l(20) = 2$

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Examples:

20 = 16 + 4 = 12 + 7 + 1

$L(20) = 3$, $l(20) = 2$

50 = 49 + 1 = 28 + 16 + 4 + 2

$L(50) = 4$, $l(50) = 2$

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$$q_{i-3} + q_{i-2} = q_i$$

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The shortest possible game on n is completed in $n - L(n)$ moves.

Proof Sketch: Strong induction on n.
If n is in the Fibonacci Quilt Sequence, denoted q_i

$$q_{i-3} + q_{i-2} = q_i$$

If n is not in the sequence

$$n = q_{i_1} + q_{i_2} + \cdots + q_{i_{L(n)}}$$

Number of moves:

$$(q_{i_1} - 1) + (q_{i_2} - 1) + \cdots + (q_{i_{L(n)}} - 1)$$
$$= (q_{i_1} + q_{i_2} + \cdots + q_{i_{L(n)}}) - L(n)$$
$$=$$

Distribution of Game Lengths

Conjecture

The distribution of a random game length approaches Gaussian as *n* increases.

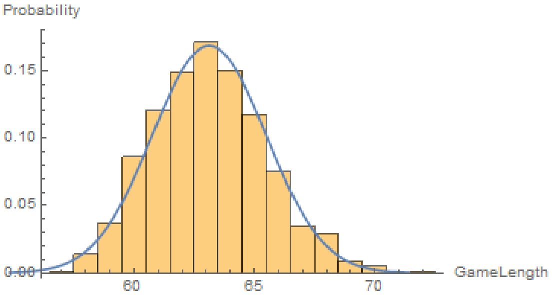

Figure: Distribution of 1000 games on n=60

Future Work

- Is there a deterministic game that always results in the lower bound?
- What patterns emerge from the winner of certain deterministic games as n increases?
- Does either player have a winning strategy?
  - Analogous result on the Zeckendorf Game shows that for $n > 2$, player 2 has a winning strategy

References

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Thank You

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