ABBA and the Random Matrix Discotheque

Keller L. Blackwell and Wanqiao Xu

SMALL REU 2019

Williams College

Joint work with Neelima Borade, Charles Devlin VI, Renyuan Ma,

Leticia Mattos da Silva, and Dr. Steven Miller

July 26th, 2019

Random Matrix Theory in Quantum Physics

Classical Mechanics: 3-body problem - intractable!

Random Matrix Theory in Quantum Physics

Classical Mechanics: 3-body problem - intractable!

Heavy Nuclei: Uranium 200+ protons/neutrons - worse!

Random Matrix Theory in Quantum Physics

Classical Mechanics: 3-body problem - intractable!

Heavy Nuclei: Uranium 200+ protons/neutrons - worse!

Schrödinger Equation

$$H\psi_n = E_n\psi_n$$

H: Hamiltonian matrix; entries dependent on system

Random Matrix Theory in Quantum Physics

Classical Mechanics: 3-body problem - intractable!

Heavy Nuclei: Uranium 200+ protons/neutrons - worse!

Schrödinger Equation

$$H\psi_n = E_n\psi_n$$

H: Hamiltonian matrix; entries dependent on system

Wigner's Insights

- Treat H as random matrix

Random Matrix Theory in Quantum Physics

Classical Mechanics: 3-body problem - intractable!

Heavy Nuclei: Uranium 200+ protons/neutrons - worse!

Schrödinger Equation

$$H\psi_n = E_n\psi_n$$

H: Hamiltonian matrix; entries dependent on system

Wigner's Insights

- Treat H as random matrix
- Eigenvalue behavior of H well approximated by averaging over ensemble

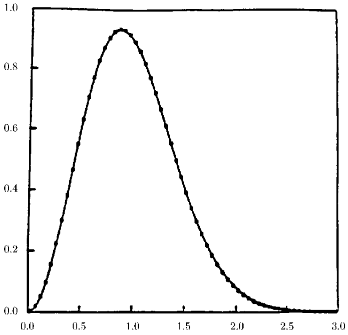
RMT and Number Theory

Montgomery's Pair Correlation Conjecture:

Spacing between Riemann-zeta function zeros: $1 - \left(\frac{\sin \pi x}{\pi x}\right)^2 + \delta(u)$

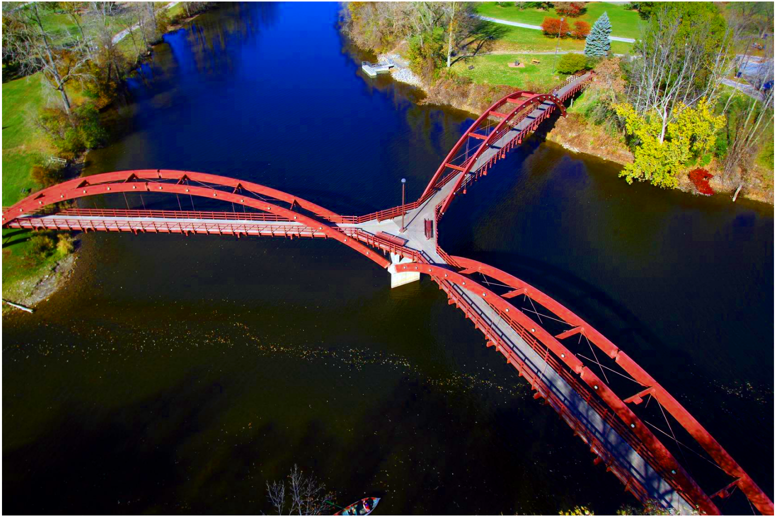
Montgomery's Pair Correlation Conjecture:
Spacing between Riemann-zeta function zeros: $1 - \left(\frac{\sin \pi x}{\pi x}\right)^2 + \delta(u)$

Montgomery's GUE Conjecture:



70 million $\zeta(s)$ zero spacings, vs. GUE prediction (Odlyzko)

Why combine random matrices?



"Tridge" in Midland, Michigan

Why combine random matrices?

	Number Theory	Random Matrix Theory
Object	L-functions	Random Matrices
Events	Zeros	Eigenvalues
Process	Rankin-Selberg Convolution	???

Setting the Stage

Component Matrices A, B

Let A, B be $N \times N$ random real symmetric matrices, with entries i.i.d. from a distribution with mean 0, variance 1, and all finite moments.

Setting the Stage

Component Matrices A, B

Let A, B be $N \times N$ random real symmetric matrices, with entries i.i.d. from a distribution with mean 0, variance 1, and all finite moments.

Imposed structure on A:

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_2 & a_1 & a_0 \\ a_1 & a_0 & a_1 & \cdots & a_3 & a_2 & a_1 \\ a_2 & a_1 & a_0 & \cdots & a_4 & a_3 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 & \cdots & a_1 & a_0 & a_1 \\ a_0 & a_1 & a_2 & \cdots & a_2 & a_1 & a_0 \end{bmatrix}$$

Symmetric Palindromic Toeplitz (SPT)

Setting the Stage

Component Matrices A, B

Let A, B be $N \times N$ random real symmetric matrices, with entries i.i.d. from a distribution p with mean 0, variance 1, and all finite moments.

Imposed structure on A:

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_3 & a_2 & a_1 & a_0 \\ a_1 & a_0 & a_1 & a_2 & a_3 & a_3 & a_2 & a_1 \\ a_2 & a_1 & a_0 & a_1 & a_2 & a_3 & a_3 & a_2 \\ a_3 & a_2 & a_1 & a_0 & a_1 & a_2 & a_3 & a_3 \\ a_3 & a_3 & a_2 & a_1 & a_0 & a_1 & a_2 & a_3 \\ a_2 & a_3 & a_3 & a_2 & a_1 & a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 & a_3 & a_2 & a_1 & a_0 & a_1 \\ a_0 & a_1 & a_2 & a_3 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}$$

Symmetric Palindromic Toeplitz (SPT)

Component Matrices A, B

Let A, B be $N \times N$ random real symmetric matrices, with entries i.i.d. from a distribution p with mean 0, variance 1, and all finite moments.

Imposed structure on A:

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_3 & a_2 & a_1 & a_0 \\ a_1 & a_0 & a_1 & a_2 & a_3 & a_3 & a_2 & a_1 \\ a_2 & a_1 & a_0 & a_1 & a_2 & a_3 & a_3 & a_2 \\ a_3 & a_2 & a_1 & a_0 & a_1 & a_2 & a_3 & a_3 \\ a_3 & a_3 & a_2 & a_1 & a_0 & a_1 & a_2 & a_3 \\ a_2 & a_3 & a_3 & a_2 & a_1 & a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 & a_3 & a_2 & a_1 & a_0 & a_1 \\ a_0 & a_1 & a_2 & a_3 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}$$

Symmetric Palindromic Toeplitz (SPT)

Disco!

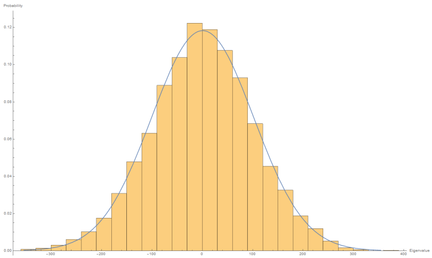
Definition: "Disco" of A, B

$$\mathcal{D}(A,B) = \begin{bmatrix} A & B \\ B & A \end{bmatrix} =$$

Limiting Distributions of A, B

A: Gaussian (Massey, Miller, and Sinsheimer, 2007)

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

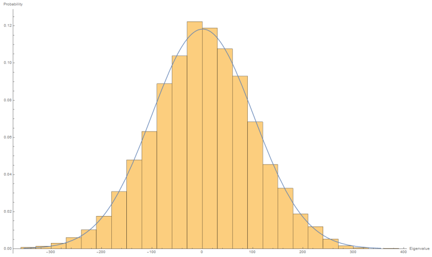


10K × 10K SPT

Limiting Distributions of A, B

A: Gaussian (Massey, Miller, and Sinsheimer, 2007)

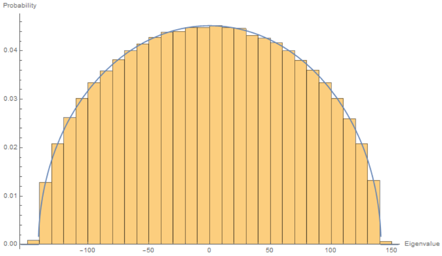
$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$



10K × 10K SPT

B: Semi-circle (Wigner, 1955)

$$f(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4 - x^2}, & |x| \le 2 \\ 0, & |x| > 2. \end{cases}$$



10K × 10K Real Symmetric

Defining Probability Space

The kth Moment of D

$$M_k(\mathcal{D}) \;=\; \lim_{N\to\infty} \frac{1}{(2N)^{\frac{k}{2}+1}} \mathbb{E}\left[\sum_{i=1}^{2N} \lambda_i^k(\mathcal{D})\right]$$

Defining Probability Space

The kth Moment of D

$$M_k(\mathcal{D}) \;=\; \lim_{N\to\infty} \frac{1}{(2N)^{\frac{k}{2}+1}} \mathbb{E}\left[\sum_{i=1}^{2N} \lambda_i^k(\mathcal{D})\right]$$

Eigenvalue Trace Lemma

$$\sum_{i=1}^{2N} \lambda_i^k(\mathcal{D}) \;=\; \text{Trace}(\mathcal{D}^k)$$

Defining Probability Space

The kth Moment of D

$$M_k(\mathcal{D}) = \lim_{N\to\infty} \frac{1}{(2N)^{\frac{k}{2}+1}} \mathbb{E}\left[\sum_{i=1}^{2N} \lambda_i^k(\mathcal{D})\right]$$

Eigenvalue Trace Lemma

$$\sum_{i=1}^{2N} \lambda_i^k(\mathcal{D}) = \text{Trace}(\mathcal{D}^k)$$

Key Equation

$$M_k(\mathcal{D}) = \lim_{N\to\infty} \frac{1}{(2N)^{\frac{k}{2}+1}} \mathbb{E}\left[\text{Trace}(\mathcal{D}^k)\right]$$

Method of Moments

Lemma

$$\begin{aligned}\operatorname{tr}\left(\mathcal{D}^k\right) &= \operatorname{tr}\left((A+B)^k\right) + \operatorname{tr}\left((A-B)^k\right) \\ &= 2\sum_{\substack{l=0 \\ l:\text{even}}}^{k} \sum_{\substack{i_1+\cdots+i_p=k-l \\ j_1+\cdots+j_p=l}} \operatorname{tr}\left(A^{i_1}B^{j_1}\cdots A^{i_p}B^{j_p}\right)\end{aligned}$$

Note: Only the terms with even power survive!

A Simple Example

Take $k = 4$.

$$\begin{aligned} M_4(\mathcal{D}) &= \lim_{N\to\infty} \frac{1}{(2N)^3}\mathbb{E}[\mathrm{tr}\left(\mathcal{D}^4\right)] \\ &= \lim_{N\to\infty} \frac{1}{(2N)^3}\mathbb{E}[\mathrm{tr}\left((A+B)^4\right) + \mathrm{tr}\left((A-B)^4\right)] \\ &= \lim_{N\to\infty} \frac{2}{(2N)^3}\mathbb{E}[\mathrm{tr}\left(A^4\right) + 4\,\mathrm{tr}\left(A^2B^2\right) + 2\,\mathrm{tr}(ABAB) + \mathrm{tr}\left(B^4\right)] \end{aligned}$$

A Simple Example

Take $k = 4$.

$$\begin{aligned} M_4(\mathcal{D}) &= \lim_{N\to\infty} \frac{1}{(2N)^3}\mathbb{E}[\mathrm{tr}\left(\mathcal{D}^4\right)] \\ &= \lim_{N\to\infty} \frac{1}{(2N)^3}\mathbb{E}[\mathrm{tr}\left((A+B)^4\right) + \mathrm{tr}\left((A-B)^4\right)] \\ &= \lim_{N\to\infty} \frac{2}{(2N)^3}\mathbb{E}[\mathrm{tr}\left(A^4\right) + 4\,\mathrm{tr}\left(A^2B^2\right) + 2\,\mathrm{tr}(ABAB) + \mathrm{tr}\left(B^4\right)] \end{aligned}$$

We know

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N^3}\mathbb{E}[\mathrm{tr}\left(A^4\right)] &= M_4(A) \\ \lim_{N\to\infty} \frac{1}{N^3}\mathbb{E}[\mathrm{tr}\left(B^4\right)] &= M_4(B) \end{aligned}$$

Let's Do Some Pairing!

$$\operatorname{tr}\left(A^2B^2\right) = \sum_{1\le i_1,\dots,i_4\le 4} a_{i_1,i_2}a_{i_2,i_3}b_{i_3,i_4}b_{i_4,i_1}$$

$$\operatorname{tr}(ABAB) = \sum_{1\le i_1,\dots,i_4\le 4} a_{i_1,i_2}b_{i_2,i_3}a_{i_3,i_4}b_{i_4,i_1}$$

Let's Do Some Pairing!

$$\mathrm{tr}\left(A^2B^2\right) = \sum_{1\leq i_1,\ldots,i_4\leq 4} a_{i_1,i_2}a_{i_2,i_3}b_{i_3,i_4}b_{i_4,i_1}$$

$$\mathrm{tr}(ABAB) = \sum_{1\leq i_1,\ldots,i_4\leq 4} a_{i_1,i_2}b_{i_2,i_3}a_{i_3,i_4}b_{i_4,i_1}$$

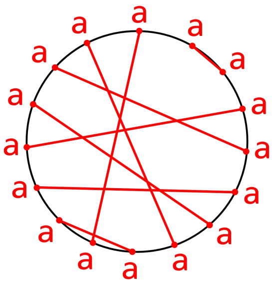
Let's Do Some Pairing!

$$\mathrm{tr}\left(A^2B^2\right) = \sum_{1\le i_1,\ldots,i_4\le 4} a_{i_1,i_2}a_{i_2,i_3}b_{i_3,i_4}b_{i_4,i_1}$$

$$\mathrm{tr}(ABAB) = \sum_{1\le i_1,\ldots,i_4\le 4} a_{i_1,i_2}b_{i_2,i_3}a_{i_3,i_4}b_{i_4,i_1}$$

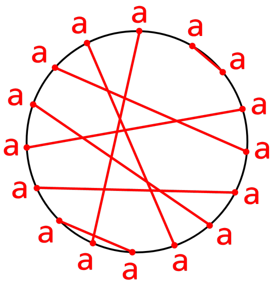
$\mathbb{E}[a] = \mathbb{E}[b] = 0 \implies a, b$ have to be paired

Gaussian

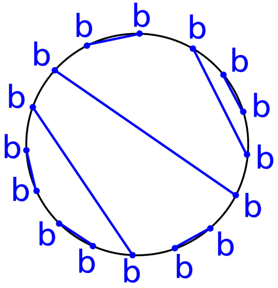


Let's Do Some Pairing!

Gaussian

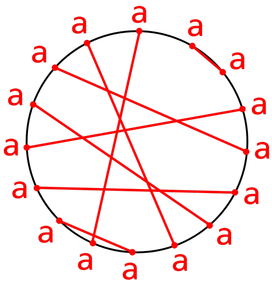


Semicircle

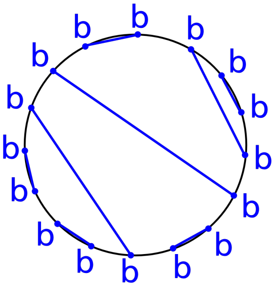


Let's Do Some Pairing!

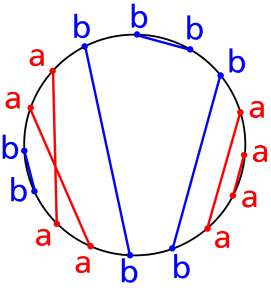
Gaussian



Semicircle



Disco



Let's Do Some Pairing!

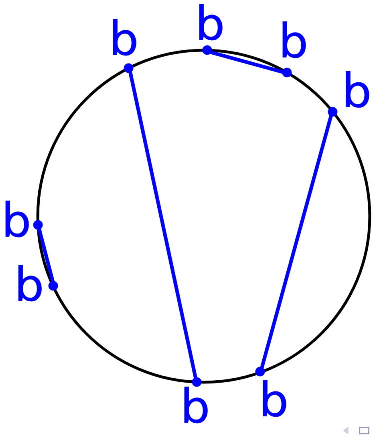
Theorem (B., Borade, Devlin, Ma, Miller, Silva, and X., 2019)

For 2α a's and 2β b's, the number of contributing pairing configurations is:

$$\mathcal{P}(\alpha,\beta) = \sum_{\substack{|V|=\beta+1 \\ deg(v)=d_1,d_2,\ldots,d_{\beta+1} \\ v\in V}} \frac{2(\alpha+\beta)}{\sigma_r(G)} \prod_{s=1}^{\beta+1} \binom{2r_s+d_s-1}{d_s-1}(2r_s-1)!!$$

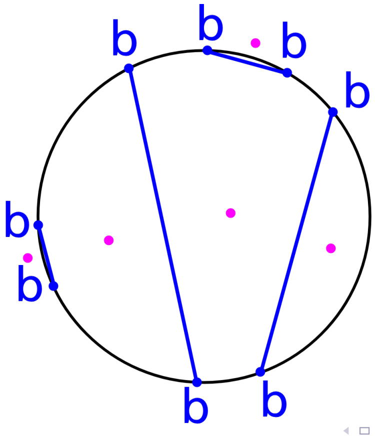
Let's Do Some Pairing!

$$\mathcal{P}(\alpha,\beta) = \sum_{\substack{|V|=\beta+1 \\ deg(v)=d_1,d_2,\ldots,d_{\beta+1} \\ v\in V}} \frac{2(\alpha+\beta)}{\sigma_r(G)} \prod_{s=1}^{\beta+1} \binom{2r_s+d_s-1}{d_s-1}(2r_s-1)!!$$



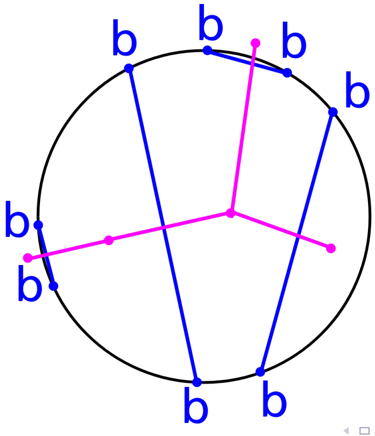
Let's Do Some Pairing!

$$\mathcal{P}(\alpha,\beta) = \sum_{\substack{|V|=\beta+1 \\ deg(v)=d_1,d_2,\ldots,d_{\beta+1} \\ v\in V}} \frac{2(\alpha+\beta)}{\sigma_r(G)} \prod_{s=1}^{\beta+1} \binom{2r_s+d_s-1}{d_s-1}(2r_s-1)!!$$



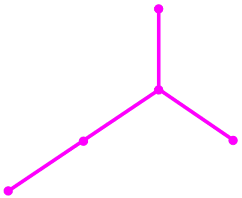
Let's Do Some Pairing!

$$\mathcal{P}(\alpha,\beta) = \sum_{\substack{|V|=\beta+1 \\ deg(v)=d_1,d_2,\ldots,d_{\beta+1} \\ v\in V}} \frac{2(\alpha+\beta)}{\sigma_r(G)} \prod_{s=1}^{\beta+1} \binom{2r_s+d_s-1}{d_s-1} (2r_s-1)!!$$



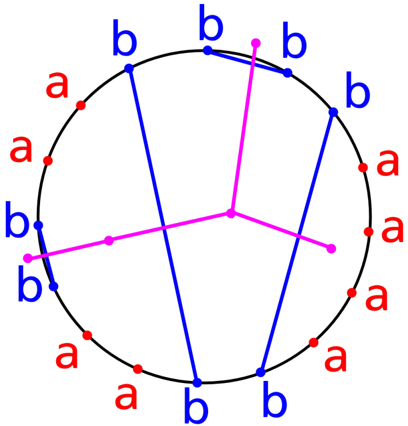
Let's Do Some Pairing!

$$\mathcal{P}(\alpha,\beta) = \sum_{\substack{|V|=\beta+1 \\ deg(v)=d_1,d_2,\ldots,d_{\beta+1} \\ v\in V}} \frac{2(\alpha+\beta)}{\sigma_r(G)} \prod_{s=1}^{\beta+1} \binom{2r_s+d_s-1}{d_s-1}(2r_s-1)!!$$



Let's Do Some Pairing!

$$\mathcal{P}(\alpha,\beta) = \sum_{\substack{|V|=\beta+1 \\ deg(v)=d_1,d_2,\ldots,d_{\beta+1} \\ v\in V}} \frac{2(\alpha+\beta)}{\sigma_r(G)} \prod_{s=1}^{\beta+1} \binom{2r_s+d_s-1}{d_s-1}(2r_s-1)!!$$



Odd and Small Even Moments

- For odd k, $M_k(\mathcal{D}) = 0$.

Odd and Small Even Moments

- For odd k, $M_k(\mathcal{D}) = 0$.
- Brute force computation of small even moments:

	Semicircle	**Disco of A, B**	**Gaussian**
2	1	1	1
4	2	2.25	3
6	5	7	15
8	14	27.5	105

Upper Bound on Even Moments

Theorem (B., Borade, Devlin, Ma, Miller, Silva, and X., 2019)

$$M_{2k}(\mathcal{D}) \leq \frac{(2k-1)!! + \sum_{j=1}^{k-1} \binom{2k}{2j}\binom{2j}{j}\frac{(2k-2j-1)!!}{j+1} + \frac{1}{k+1}\binom{2k}{k}}{2^{k-1}}$$
$$\leq M_{2k}(A)$$

Upper Bound on Even Moments

Theorem (B., Borade, Devlin, Ma, Miller, Silva, and X., 2019)

$$M_{2k}(\mathcal{D}) \leq \frac{(2k-1)!! + \sum_{j=1}^{k-1} \binom{2k}{2j}\binom{2j}{j}\frac{(2k-2j-1)!!}{j+1} + \frac{1}{k+1}\binom{2k}{k}}{2^{k-1}}$$
$$\leq M_{2k}(A)$$

Sketch:

$$(2k-1)!! \quad \leftrightarrow \quad \text{Contribution of } \mathbb{E}\left[\mathrm{Tr}\left(A^{2k}\right)\right]$$

$$\frac{1}{k+1}\binom{2k}{k} \quad \leftrightarrow \quad \text{Contribution of } \mathbb{E}\left[\mathrm{Tr}\left(B^{2k}\right)\right]$$

$$\binom{2j}{j}\frac{(2k-2j-1)!!}{j+1} \quad \leftrightarrow \quad \text{Contribution of } \mathbb{E}\left[\mathrm{Tr}\left(A^{2k-2j}B^{2j}\right)\right]$$

Lower Bound on Even Moments

Theorem (B., Borade, Devlin, Ma, Miller, Silva, and X., 2019)

$$M_{2k}(\mathcal{D}) \geq \frac{(2k-1)!! + \sum_{j=1}^{k-1} 2k\binom{2j}{j}\frac{(2k-2j-1)!!}{j+1} + \frac{1}{k+1}\binom{2k}{k}}{2^{k-1}}$$

$$\geq M_{2k}(B)$$

Lower Bound on Even Moments

Theorem (B., Borade, Devlin, Ma, Miller, Silva, and X., 2019)

$$M_{2k}(\mathcal{D}) \geq \frac{(2k-1)!! + \sum_{j=1}^{k-1} 2k\binom{2j}{j}\frac{(2k-2j-1)!!}{j+1} + \frac{1}{k+1}\binom{2k}{k}}{2^{k-1}}$$

$$\geq M_{2k}(B)$$

Sketch:

$$(2k-1)!! \quad \leftrightarrow \quad \text{Contribution of } \mathbb{E}\left[\mathrm{Tr}\left(A^{2k}\right)\right]$$

$$\frac{1}{k+1}\binom{2k}{k} \quad \leftrightarrow \quad \text{Contribution of } \mathbb{E}\left[\mathrm{Tr}\left(B^{2k}\right)\right]$$

$$\binom{2j}{j}\frac{(2k-2j-1)!!}{j+1} \quad \leftrightarrow \quad \text{Contribution of } \mathbb{E}\left[\mathrm{Tr}\left(A^{2k-2j}B^{2j}\right)\right]$$

Limiting Bounds

Theorem (B., Borade, Devlin, Ma, Miller, Silva, and X., 2019)

$$\lim_{k\to\infty} \frac{M_{2k}(\mathcal{D})}{M_{2k}(A)} = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \frac{M_{2k}(B)}{M_{2k}(\mathcal{D})} = 0.$$

Limiting Bounds

Theorem (B., Borade, Devlin, Ma, Miller, Silva, and X., 2019)

$$\lim_{k\to\infty} \frac{M_{2k}(\mathcal{D})}{M_{2k}(A)} = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \frac{M_{2k}(B)}{M_{2k}(\mathcal{D})} = 0.$$

Sketch:

$$\left| \frac{(2k-1)!! + \sum_{j=1}^{k-1} \binom{2k}{2j}\binom{2j}{j}\frac{(2k-2j-1)!!}{j+1} + \frac{1}{k+1}\binom{2k}{k}}{2^{k-1}(2k-1)!!} \right| \leq \mathcal{O}\left(\left(\frac{5}{6}\right)^k\right)$$

Limiting Bounds

Theorem (B., Borade, Devlin, Ma, Miller, Silva, and X., 2019)

$$\lim_{k\to\infty} \frac{M_{2k}(\mathcal{D})}{M_{2k}(A)} = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \frac{M_{2k}(B)}{M_{2k}(\mathcal{D})} = 0.$$

Sketch:

$$\left| \frac{(2k-1)!! + \sum_{j=1}^{k-1} \binom{2k}{2j}\binom{2j}{j}\frac{(2k-2j-1)!!}{j+1} + \frac{1}{k+1}\binom{2k}{k}}{2^{k-1}(2k-1)!!} \right| \leq \mathcal{O}\left(\left(\frac{5}{6}\right)^k\right)$$

$$\lim_{k\to\infty} \frac{2^{k-1}\frac{1}{k+1}\binom{2k}{k}}{(2k-1)!! + \sum_{j=1}^{k-1} 2k\binom{2j}{j}\frac{(2k-2j-1)!!}{j+1} + \frac{1}{k+1}\binom{2k}{k}} = 0$$

Unbounded Support

Theorem (B., Borade, Devlin, Ma, Miller, Silva, and X., 2019)

The limiting eigenvalue distribution P of $\mathcal{D}(A, B)$ has unbounded support.

Unbounded Support

Theorem (B., Borade, Devlin, Ma, Miller, Silva, and X., 2019)

The limiting eigenvalue distribution P of $\mathcal{D}(A, B)$ has unbounded support.

Sketch:

If $\text{supp}(P) \subset [-B, B]$ then $\sqrt[2k]{M_{2k}(\mathcal{D})} \leq B$ for all k.

Unbounded Support

Theorem (B., Borade, Devlin, Ma, Miller, Silva, and X., 2019)

The limiting eigenvalue distribution P of $\mathcal{D}(A, B)$ has unbounded support.

Sketch:

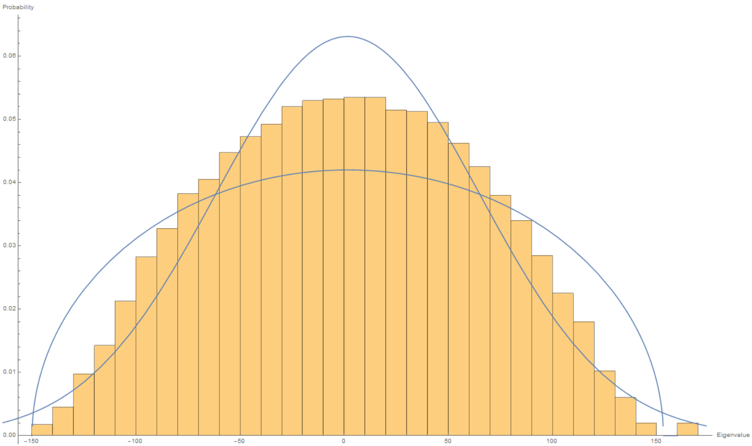
If $\operatorname{supp}(P) \subset [-B, B]$ then $\sqrt[2k]{M_{2k}(\mathcal{D})} \leq B$ for all k.

$$\sqrt[2k]{M_{2k}(\mathcal{D})} \geq \left[\frac{(2k-1)!! + \sum_{j=1}^{k-1} 2k\binom{2j}{j}\frac{(2k-2j-1)!!}{j+1} + \frac{1}{k+1}\binom{2k}{k}}{2^k}\right]^{\frac{1}{2k}}$$

$$> \frac{1}{2}\sqrt{k} \qquad \textit{(Stirling's Formula)}$$

$$\gg B$$

Disco vs. Gaussian, Semicircle



Thank you!