Extensions of Autocorrelation Inequalities with Applications to Additive Combinatorics

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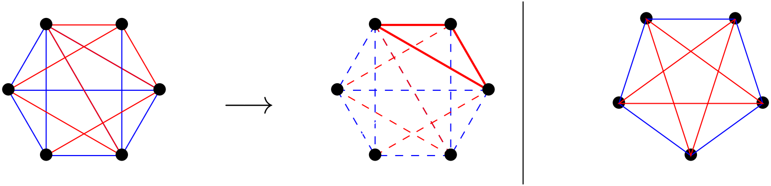
Discrete Ramsey theory

Question

How large must a discrete structure be in order for a substructure to appear?

Example (Ramsey's Theorem)

Given $r, b \in \mathbb{N}$. How large must a 2-colored copy of K_n be in order for a red copy of K_r or a blue copy of K_b to appear?



The above example demonstrates $R(3, 3) = 6$.

Discrete Ramsey theory: rephrasing the question

Question, original

How large must a discrete structure be in order for a substructure to appear?

Question, rephrased

Given a discrete structure of fixed size, what is the most structured substructure that must appear?

Example

Given $n \in \mathbb{N}$. Given a 2-colored copy of K_n, what are the largest $r, b \in \mathbb{N}$ such that a red copy of K_r or a blue copy of K_b must appear?

Continuous Ramsey theory

Ramsey-type arguments also apply in a continuous setting.

Question

Given a continuous structure of fixed size, what is the most structured substructure that is guaranteed to appear?

Example

Fix a nonnegative $f \in L^1(\mathbb{R})$. Its **autocorrelation** is

$$g(t) := \int_{\mathbb{R}} f(x)f(x+t)\,dx.$$

Given the autocorrelation $g(t)$, for some fixed $||f||_{L^1}$, what is the $t \in [0,1]$ so that $g(t)$ is minimized?

Preliminaries

Definition

Let $f : \mathbb{R} \to \mathbb{R}$. The L^1 and L^2 norms of f are defined by

$$||f||_{L^1} = \int_{\mathbb{R}} |f(x)|\, dx\ , \qquad ||f||_{L^2} = \left(\int_{\mathbb{R}} (f(x))^2\, dx \right)^{1/2}$$

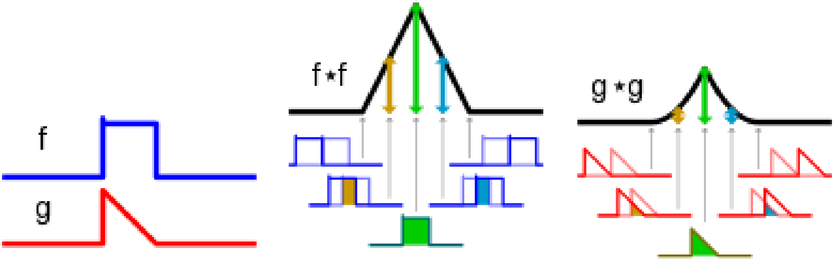
If $||f||_{L^1} < \infty$, we write $f \in L^1(\mathbb{R})$.

Autocorrelation – intuition

Definition

Fix a nonnegative $f \in L^1(\mathbb{R})$. Its **autocorrelation** is

$$g(t) = \int_{\mathbb{R}} f(x)f(x+t)\,dx.$$



Properties of the autocorrelation

Definition

Fix a nonnegative $f \in L^1(\mathbb{R})$. Its **autocorrelation** is

$$g(t) = \int_{\mathbb{R}} f(x)f(x+t)\,dx.$$

Standard techniques quickly give bounds on $g(t)$.

- **Lower bound:** Since $f \geq 0$, we have $g \geq 0$.
- **Upper bound:** By Cauchy-Schwarz,

$$g(t) = \int_{\mathbb{R}} f(x)f(x+t)\,dx \leq \int_{\mathbb{R}} (f(x))^2\,dx = ||f||_{L^2}^2.$$

Is the upper bound sharp?

We just showed the upper bound

$$g(t) = \int_{\mathbb{R}} f(x)f(x+t)\ dx \leq ||f||_{L^2}^2.$$

Question: Is this sharp?

Answer: Yes! For $t = 0$, we have

$$g(0) = \int_{\mathbb{R}} f(x)f(x+0)\ dx = ||f||_{L^2}.$$

Is the lower bound sharp?

We have the upper bound

$$g(t) = \int_{\mathbb{R}} f(x)f(x+t)\,dx \geq 0.$$

Question: Is this sharp?

Answer: No, and finding a sharp lower bound is difficult.

Finding a sharp lower bound is equivalent to finding

$$\min_t g(t).$$

Lower bound question: useless answers

To make this problem well-posed, we have to normalize.

Useless answer #1

Pick t very large. Then

$$\min_t g(t) = \min_t \int_{\mathbb{R}} f(x)f(x+t)\, dx \approx 0.$$

Useless answer #2

Let $f = \varepsilon \cdot f_2$ for some $\epsilon > 0$. Then

$$\min_t g(t) = \min_t \int_{\mathbb{R}} f(x)f(x+t)\, dx = \varepsilon^2 \int_{\mathbb{R}} f_2(x)f_2(x+t)\, dx.$$

As $\varepsilon \to 0$, $\min_t g(t) = \min_t \int_{\mathbb{R}} f(x)f(x+t)\, dx \to 0$.

Fixing useless answer #1

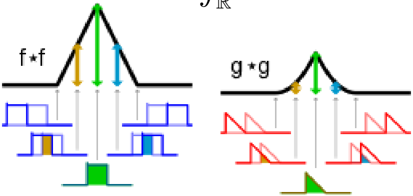
Useless answer #1

Pick t very large. Then

$$\min_t g(t) = \min_t \int_{\mathbb{R}} f(x)f(x+t)\,dx \approx 0.$$

To fix #1, we instead consider

$$\min_{t\in[0,1]} g(t) = \min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx.$$



Fixing useless answer #2

Useless answer #2

Let $f = \varepsilon \cdot f_2$ for some $\epsilon > 0$. Then

$$\min_{t\in[0,1]} g(t) = \min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx = \min_{t\in[0,1]} \varepsilon^2 \int_{\mathbb{R}} f_2(x)f_2(x+t)\,dx.$$

As $\varepsilon \to 0$,

$$\min_{t\in[0,1]} g(t) = \min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx \to 0.$$

To fix #2, because $f \in L^1(\mathbb{R})$, we instead consider

$$\min_{t\in[0,1]} \frac{g(t)}{||f||_{L^1}^2} = \min_{t\in[0,1]} \frac{\int_{\mathbb{R}} f(x)f(x+t)\,dx}{||f||_{L^1}^2}.$$

Lower bound question: rephrased

Question

What is the smallest $c > 0$ such that for any choice of f

$$\min_{t\in[0,1]} g(t) = \min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx \le c||f||_{L^1}^2?$$

Using standard techniques, we can show

$$\min_{t\in[0,1]} g(t) = \min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx \le 0.5||f||_{L^1}^2.$$

Lower bound question: 0.5 bound

Because the minimum is less than or equal to the average,

$$\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx \leq \int_0^1 \int_{\mathbb{R}} f(x)f(x+t)\,dx\,dt.$$

By symmetry,

$$\int_0^1 \int_{\mathbb{R}} f(x)f(x+t)\,dx\,dt = \int_{-1}^0 \int_{\mathbb{R}} f(x)f(x+t)\,dx\,dt,$$

so

$$\int_0^1 \int_{\mathbb{R}} f(x)f(x+t)\,dx\,dt = 0.5 \int_{-1}^1 \int_{\mathbb{R}} f(x)f(x+t)\,dx\,dt.$$

Lower bound question: 0.5 bound

From the previous slide, we have

$$\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx \leq 0.5 \int_{-1}^{1}\int_{\mathbb{R}} f(x)f(x+t)\,dx\,dt.$$

Now we estimate

$$\begin{aligned}\int_{-1}^{1}\int_{\mathbb{R}} f(x)f(x+t)\,dx\,dt &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)f(x+t)\,dx\,dt \\ \text{(by Fubini)} \quad &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)f(x+t)\,dt\,dx \\ &= \int_{\mathbb{R}} f(x)\left(\int_{\mathbb{R}} f(x+t)\,dt\right)\,dx \\ &= ||f||_{L^1}^2.\end{aligned}$$

Lower bound question: 0.5 bound

From the previous slides, we have

$$\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx \le 0.5 \int_{-1}^{1}\int_{\mathbb{R}} f(x)f(x+t)\,dx\,dt.$$

$$\int_{-1}^{1}\int_{\mathbb{R}} f(x)f(x+t)\,dx\,dt \le ||f||_{L^1}^2$$

Putting these together gives

$$\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx \le 0.5||f||_{L^1}^2.$$

An improved lower bound

We just showed

$$\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx \le 0.5||f||_{L^1}^2.$$

With more advanced techniques, the following holds.

Theorem (Barnard and Steinerberger, 2019)

Let $f \in L^1(\mathbb{R})$. Then

$$\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx \le 0.411||f||_{L^1}^2,$$

and the constant 0.411 cannot be replaced by .37.

An improved lower bound

Theorem (Barnard and Steinerberger, 2019)

Let $f \in L^1(\mathbb{R})$. Then

$$\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx \leq 0.411||f||_{L^1}^2,$$

and the constant 0.411 cannot be replaced by .37.

Proof sketch

0.411 bound: Fourier analysis.
0.37 bound: Providing an explicit f for which

$$\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx = 0.37||f||_{L^1}^2.$$

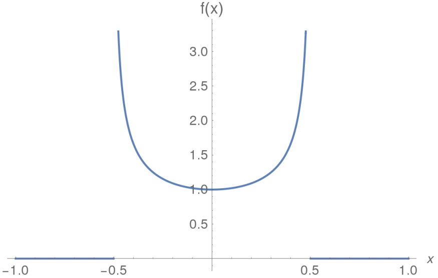
0.31 bound

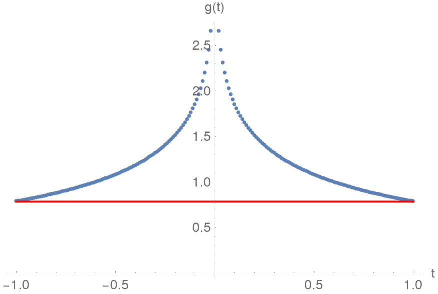
The function

$$f(x) = \frac{\chi_{[-0.5,0.5]}(x)}{\sqrt{1-4x^2}}$$

satisfies

$$\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx = 0.31||f||_{L^1}^2.$$





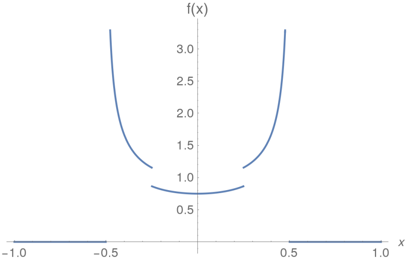
0.37 bound

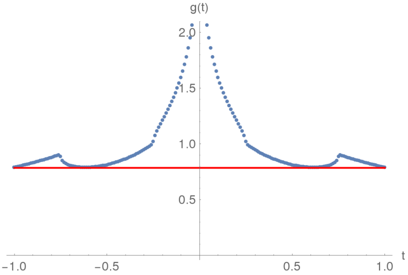
The function

$$f(x) = \frac{\chi_{[-0.5,0.5]}(x)}{\sqrt{1-4x^2}} - \frac{1}{4}\frac{\chi_{[-0.25,0.25]}(x)}{\sqrt{1-4x^2}}$$

satisfies

$$\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx = 0.37||f||_{L^1}^2.$$





Our Question

Recall the following:

Theorem (Barnard and Steinerberger, 2019)

For all $f \in L^1(\mathbb{R})$,

$$\frac{\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\, dx}{||f||_{L^1}^2} \leq 0.411,$$

and this ratio can be as large as 0.37.

Question: What can we say about the structure of functions f which maximize this ratio?

The Main Result

Theorem (F., K., Miller 2019)

A continuous function $f \in \mathbf{L}^1(\mathbb{R})$ maximizing

$$\frac{\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx}{||f||_{L^1}^2}$$

must satisfy the following two inequalities:

$$\max_{x_1\in\mathbb{R}} \min_{t\in[0,1]} f(x_1-t)+f(x_1+t) \leq \frac{2}{||f||_{L^1}} \min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)dx,$$

$$\max_{x_1\in\mathbb{R}} \min_{t\in[0,1]} f(x_1-t)+f(x_1+t) \leq \min_{x_2\in supp(f)} \max_{t\in[0,1]} f(x_2-t)+f(x_2+t).$$

The Main Approach

Given some f, can we add some small g to improve the ratio in question? That is,

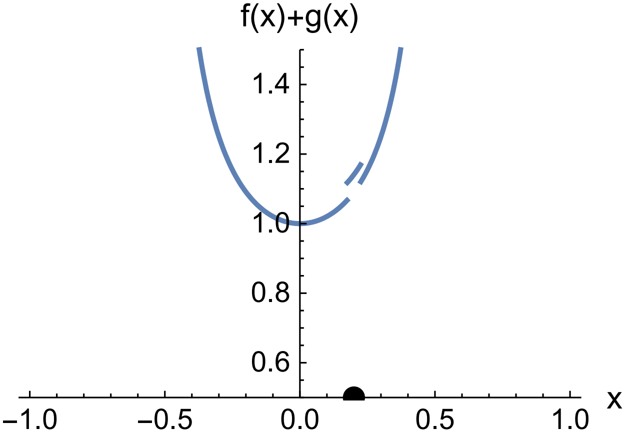
$$\frac{\min\limits_{t\in[0,1]} \int_{\mathbb{R}} (f+g)(x)(f+g)(x+t)dx}{||f+g||_{L^1}^2} > \frac{\min\limits_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)dx}{||f||_{L^1}^2}.$$

- A natural choice of g; for $\varepsilon > 0$ and $x_1 \in \mathbb{R}$, set

$$g = \varepsilon \chi_{[x_1 - \frac{\varepsilon}{2}, x_1 + \frac{\varepsilon}{2}]}.$$

Perturbation

If $g = \varepsilon\chi_{[x_1 - \frac{\varepsilon}{2}, x_1 + \frac{\varepsilon}{2}]}$, $f + g$ looks like



Perturbation

We can determine sufficient conditions for an increase.

$$\frac{\min_{t\in[0,1]} \int_{\mathbb{R}} (f+g)(x)(f+g)(x+t)dx}{||f+g||_{L^1}^2} \geq \frac{\min_{t\in[0,1]} \int_{\mathbb{R}} (f+g)(x)(f+g)(x+t)dx}{(||f||_{L^1} + ||g||_{L^1})^2}$$

We can break up the minimum, to isolate terms, so we must control

$$\min_{t\in[0,1]} \int_{\mathbb{R}} g(x)f(x+t) + f(x)g(x+t) + g(x)g(x+t)dx.$$

Perturbation

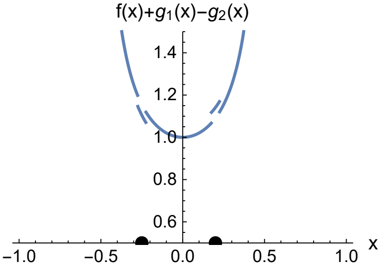
Our goal is to control

$$\min_{t\in[0,1]} \int_{\mathbb{R}} g(x)f(x+t) + f(x)g(x+t) + g(x)g(x+t)dx \approx \min_{t\in[0,1]} \varepsilon^2 \left(f(x_1+t) + f(x_1-t)\right).$$

- For small enough ε, $g(x)g(x+t) << g(x)f(x+t)$.
- Using continuity of f, and $g(x) \neq 0$ iff $x \approx x_1$, the integral becomes a function evaluation

An Alternative Perturbation

Can we conserve mass so that the L^1 norm is unchanged?
Replace f with $f + g_1 - g_2$;



An Alternative Perturbation

Using similar analytic methods,

$$\frac{\min_{t\in[0,1]} \int_{\mathbb{R}} (f+g_1-g_2)(x)\,(f+g_1-g_2)(x+t)\,dx}{||f||_{L^1}^2} > \frac{\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx}{||f||_{L^1}^2}$$

whenever

$$\max_{x_1\in\mathbb{R}} \min_{t\in[0,1]} f(x_1-t)+f(x_1+t) > \min_{x_2\in\text{supp}(f)} \max_{t\in[0,1]} f(x_2-t)+f(x_2+t).$$

The Main Result

Theorem (F., K., Miller 2019)

A continuous function $f \in \mathbf{L^1}(\mathbb{R})$ maximizing

$$\frac{\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx}{||f||_{L^1}^2}$$

must satisfy the following two inequalities:

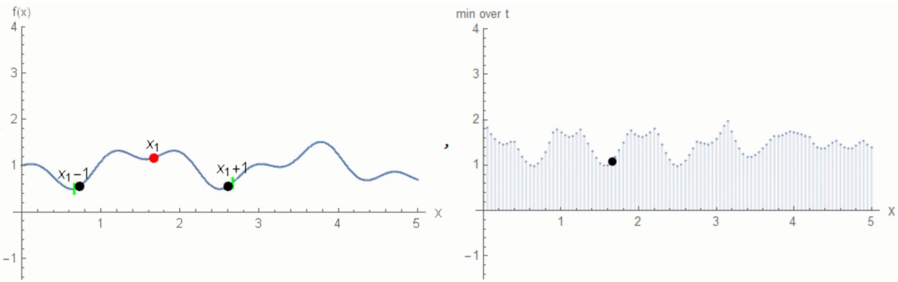
$$\max_{x_1\in\mathbb{R}} \min_{t\in[0,1]} f(x_1-t)+f(x_1+t) \leq \frac{2}{||f||_{L^1}} \min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)dx,$$

$$\max_{x_1\in\mathbb{R}} \min_{t\in[0,1]} f(x_1-t)+f(x_1+t) \leq \min_{x_2\in supp(f)} \max_{t\in[0,1]} f(x_2-t)+f(x_2+t).$$

Interpretation

What does this mean?

$$\max_{x_1 \in \mathbb{R}} \min_{t \in [0,1]} f(x_1 - t) + f(x_1 + t) \leq \min_{x_2 \in \text{supp}(f)} \max_{t \in [0,1]} f(x_2 - t) + f(x_2 + t)$$



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Generalizations

What about

$$\frac{\min_{t_1,t_2\in[0,1]}\int_{\mathbb{R}} f(x)f(x+t_1)f(x+t_2)dx}{||f||_{L^1}^3}$$

or

$$\frac{\min_{t\in[0,1]}\int_{\mathbb{R}} f(x)f(x+t)f(x+2t)dx}{||f||_{L^1}^3}$$

or even

$$\frac{\min_{t_1,t_2\in[0,1]}\int_{\mathbb{R}} f(x)f(x+t_1)f(x+t_2)f(x+t_2-t_1)dx}{||f||_{L^1}^4}?$$

Generalizations

Theorem (F., K., Miller)

Let $d \leq n$ be positive integers and A a $d \times n$ matrix. Then a continuous $f \in L^1(\mathbb{R})$ maximizing

$$\frac{\min_{\mathbf{t} \in [0,1]^d} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x + \mathbf{t} \cdot a_i) dx}{||f||_{L^1}^n}$$

must satisfy both...

Generalizations

Theorem (F., K., Miller), continued

must satisfy both...

$$\max_{x_1\in\mathbb{R}}\min_{\mathbf{t}\in[0,1]^d}\sum_{i=1}^{n}\prod_{i=1,i\neq j}^{n} f(x_1+\mathbf{t}\cdot(a_i-a_j)) \leq \frac{n}{||f||_{L^1}}\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)dx$$

and

$$\max_{x_1\in\mathbb{R}}\min_{\mathbf{t}\in[0,1]^d}\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n} f(x_1+\mathbf{t}\cdot(a_i-a_j))$$

$$\leq \min_{x_2\in\mathrm{supp}(f)}\max_{\mathbf{t}\in[0,1]^d}\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n} f(x_2+\mathbf{t}\cdot(a_i-a_j)).$$

Future Work

- Can we provide examples of good functions extremizing these generic problems?
- Can we determine the true extreme, between 0.37 and 0.411?
- Are there broad classes of functions *f* for which these conditions fail?

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Questions