Distribution of Missing Sums in Correlated Sumsets

Thomas Martinez (Harvey Mudd College, *tmartinez@hmc.edu*), Dylan King (Wake Forest University, *kingda16@wfu.edu*)
Joint research with Chenyang Sun. Advised by Steven J. Miller.
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Introduction

Considering a set $A \subseteq \{0, 1, ..., n-1\}$, we define a **sumset** as

$$A + A = \{i + j \mid i, j \in A\}.$$

We look at random sets A where $\mathbb{P}(i \in A) = p$.
Notation: $q := 1 - p$.
$M_{[0,n-1]} := |[0, 2n-2] \backslash (A+A)| = 2n-1-|A+A|$.
$m_n(k) := \mathbb{P}(M_{[0,n-1]} = k)$.
Uniform Distribution: Most previous work focused on $p = 1/2$. This implies every subset A has an equal probability of occurring.

Previous Results

Let $p = \frac{1}{2}$. Martin and O'Bryant found

$$\mathbb{E}[|A + A|] = 2n - 1 - 10 + O((3/4)^{n/2}).$$

Let $n > 5k$. Lazarev, Miller and O'Bryant found

$$2^{-k/2} \ll m_n(k) \ll (\phi/2)^k,$$

where the implied constants are independent of k and n, and ϕ is the golden ratio.

Expected Value Generalization

We generalize previous results for any p. The explicit formula for $\mathbb{E}[|A + A|]$ is stated below.

Expected Value Formula

For $p \in [0, 1]$, $\mathbb{E}[|A + A|]$ is

$$\sum_{r=0}^{n} p^r q^{n-r} \binom{n}{r} \left(2 \sum_{k=0}^{n-1} \left(1 - \frac{f(k)}{\binom{n}{r}} \right) - \left(1 - \frac{f(n-1)}{\binom{n}{r}} \right) \right),$$

where

$$f(k) = \begin{cases} \sum_{i=\frac{k+1}{2}}^{k+1} 2^{k+1-i} \binom{\frac{k+1}{2}}{i-\frac{k+1}{2}} \binom{n-k-1}{n-r-i} & \text{for } k \text{ odd} \\ \sum_{i=\frac{k}{2}}^{k} 2^{k-i} \binom{\frac{k}{2}}{i-\frac{k}{2}} \binom{n-k-1}{n-r-1-i} & \text{for } k \text{ even}. \end{cases}$$

We also find bounds for the Expected Value.

Expected Value Bounds

For $p \in [0, 1)$,

$$\mathbb{E}[|A + A|] < 2n - 1 - \frac{\sqrt{q}}{1 - \sqrt{q}}.$$

For $p \in (0.5, 1]$

$$2n - 1 - \frac{\sqrt{2q}}{1 - \sqrt{2q}} < \mathbb{E}[|A + A|].$$

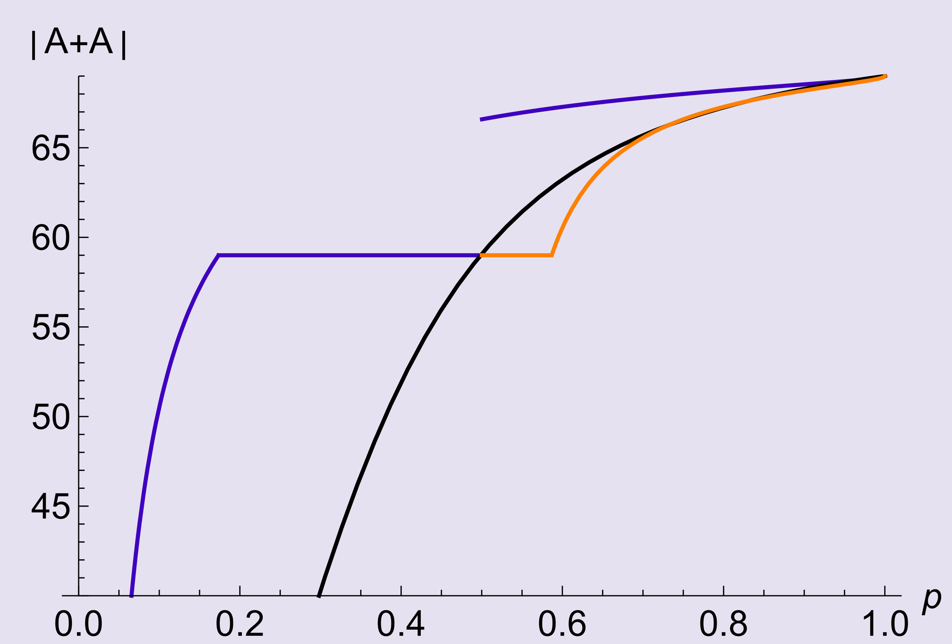


Figure 1: Above is a plot for $n = 35$.

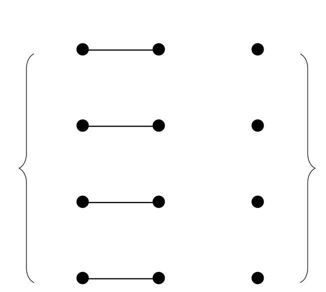
Approach

Previously, every subset had an equal chance of occurring, which allowed for a simpler calculation. Now, we must take into account the probability that $|A| = r$, where $r \in [0, n]$. Using similar techniques, we found

$$\mathbb{E}[|A+A|] = \sum_{r=0}^{n} p^r q^{n-r} \binom{n}{r} \sum_{k=0}^{2n-2} (1 - \mathbb{P}(k \notin A + A \mid |A| = r)).$$

We compute $\mathbb{P}(k \notin A + A \mid |A| = r)$ using graph theory. We define $G = (V, E)$ with vertices representing elements in $[0, n-1]$ and an edge is present if $v_1 + v_2 = k$. We are interested in finding a **vertex cover** with $n - r$ vertices.

$\frac{k+1}{2}$ disjoint edges



$n - k - 1$ isolated vertices

From this, we can derive $f(k)$ from the Expected Value formula.

$m_n(k)$ Generalization

We generalize previous results of $m_n(k)$ for any p, and show a plot for $k = 10$.

$m_n(k)$ Bounds

Let $n > \frac{2 \log(q)}{\log(1-p^2)} k$ and $\phi(p) := \sqrt{1 + 2p - 3p^2}$. Then,

$$q^{k/2} \ll m_n(k) \ll \left(\frac{1 - p + \phi(p)}{2} \right)^k.$$

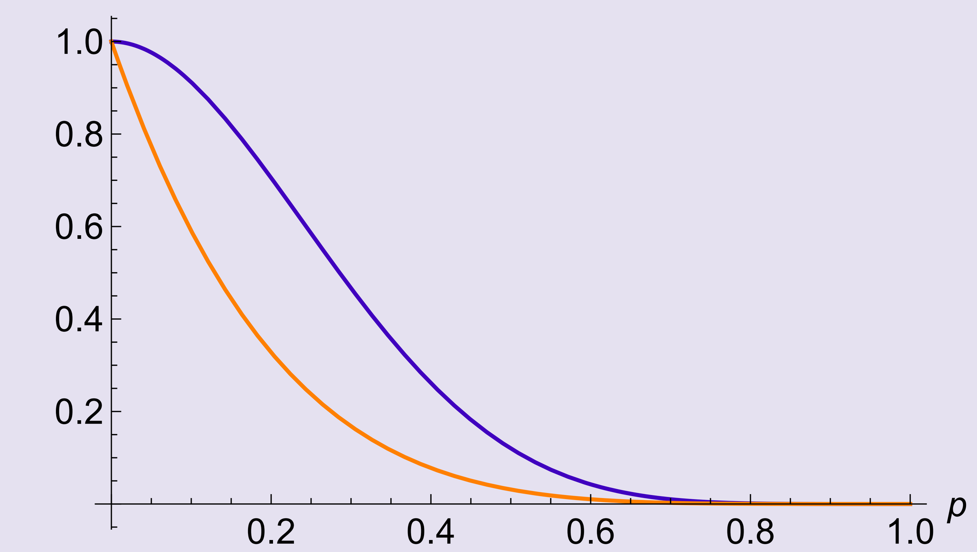


Figure 2: Above is a plot for $k = 10$.

The lower bound is achieved by finding the probability that the first $k/2$ elements are not in A, and showing that the probability that the rest of the elements in A is a subset A' such that $A' + A'$ has no missing elements (which is much more likely that the first condition). We get

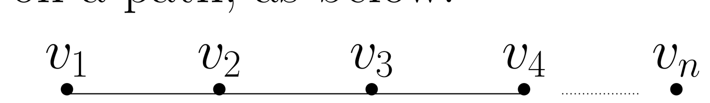
$$m_n(k) \geq \mathbb{P}(A = k/2 + A' \text{ and } M_{n-k/2}(A') = 0)$$
$$= q^{k/2}\, \mathbb{P}(M_{n-k/2}(A') = 0) \gg q^{k/2}.$$

The upper bound is achieved from noting that missing an element at least $k/2$ elements away from the ends of $[0, 2n-2]$ is very unlikely. For an upper bound, notice that missing k elements implies that missing an element at least $k/2$ elements away from the ends of $[0, 2n-2]$. This event is unlikely, because there are so many pairs of numbers that add up to an element in the middle of $A + A$, so we look at the probability of this event as our upper bound.

After some manipulation, we find

$$m_n(k) \leq \mathbb{P}(A + A \text{ misses 2 elements greater than } k - 3)$$
$$= \sum_{k-3<i<j} \mathbb{P}(i \text{ and } j \notin A + A) \ll \left(\frac{1 - p + \phi(p)}{2} \right)^k.$$

Where the last term comes from a generalization of graph theory introduced by Lazarev, Miller and O'Bryant. We seek to find $\mathbb{P}(i, j \notin A + A)$ and using a similar definition of a graph as earlier, this is the same as finding a vertex cover of missing elements on a path, as below.



Letting n denote the number of paths and a_n denoting the probability of a vertex cover, we derive $a_n = q a_{n-1} + pq a_{n-2}$, and get a closed form

$$a_n = \frac{(\phi(p) - 1 - p)(1 - p - \phi(p))^n}{2^{n+1}\, \phi(p)} + \frac{(\phi(p) + 1 + p)(1 - p + \phi(p))^n}{2^{n+1}\, \phi(p)}.$$

References

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2. O. Lazarev, S. J. Miller, K. O'Bryant, Distribution of missing sums in sumsets. Exp. Math. 22(2), 132-156 (2013) `https://arxiv.org/abs/1109.4700`

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