Distribution of Gaps in Zeckendorf Decompositions from d-dimensional Lattices

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Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$; $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$.

Zeckendorf Decomposition

Every positive integer can be written in a unique way as a sum of non-consecutive Fibonacci numbers.

A **simple jump path** is a path on the lattice grid where each movement (referred to as **step**) on the lattice grid consists of at least one unit movement to the left and one unit movement downward.

84	⋯	⋯	⋯	⋯	⋯	⋯	⋯
50	82	⋯	⋯	⋯	⋯	⋯	⋯
28	48	74	⋯	⋯	⋯	⋯	⋯
14	24	40	66	⋯	⋯	⋯	⋯
7	12	20	33	59	⋯	⋯	⋯
3	5	9	17	30	56	⋯	⋯
1	2	4	8	16	29	54	⋯

Notation

$\mathbf{s_{a,b}}$: # simple jump paths $(a,b) \to (0,0)$.
$\mathbf{t_{a,b}(k)}$: # simple jump paths $(a,b) \to (0,0)$ with k steps.

$$s_{a,b} = \sum_{k=1}^{a} t_{a,b}(k) = \sum_{k=1}^{a} \binom{a-1}{k-1}\binom{b-1}{k-1}.$$

Gaussianity of Summands

The distribution of the number of summands among all simple jump paths with starting point (i,j) where $1 \le i,j \le n$ converges to a Gaussian as $n \to \infty$.

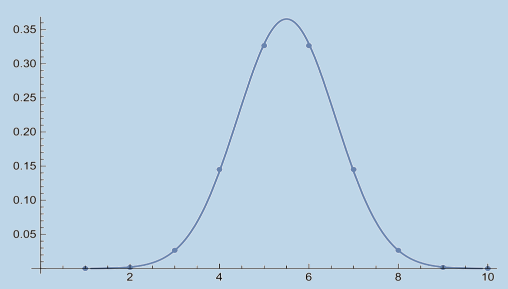

Figure 1: Distribution of the number of simple jump paths starting at $(10,10)$ versus the best fit Gaussian.

Acknowledgements

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New Results

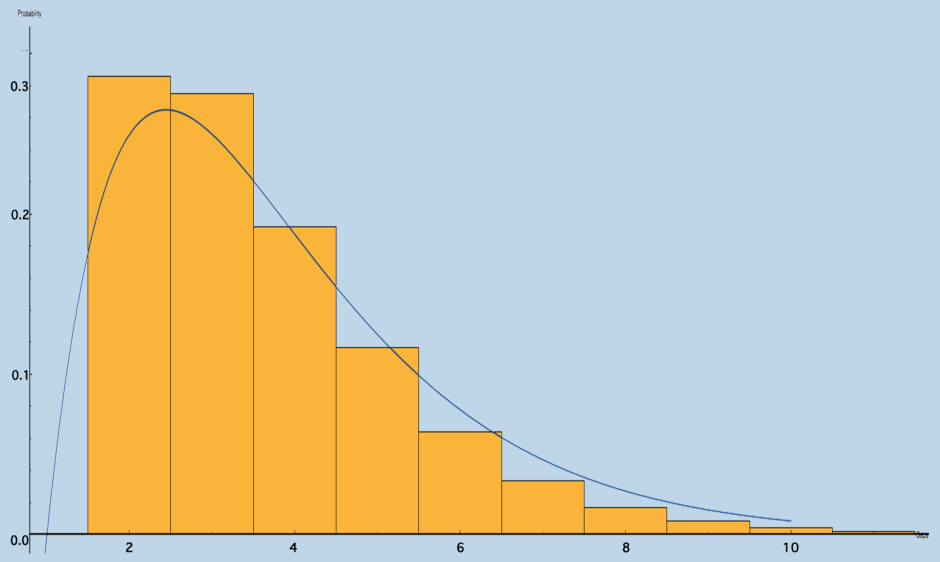

Figure 2: Distribution of gap sums starting at $(10,10)$ versus $f(k) = (k-1)/2^k$.

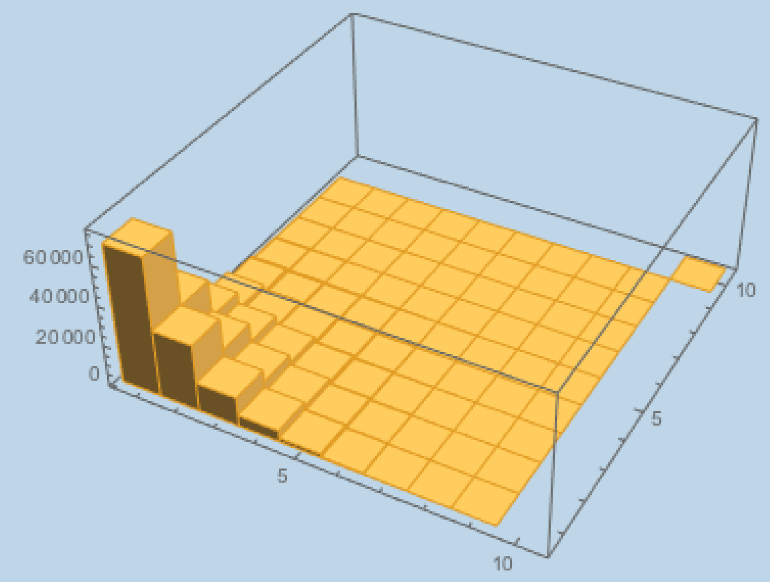

Figure 3: Distribution of gap vectors starting at $(10,10)$.

A **gap vector** of a step from $(x_{m,1}, x_{m,2})$ to $(x_{m+1,1}, x_{m+1,2})$ is the difference vector $(x_{m,1} - x_{m+1,1}, x_{m,2} - x_{m+1,2})$, and a **gap sum** is the sum of components of a gap vector.
$\mathbf{g_{a,b}}$: # gaps of simple jump paths $(a,b) \to (0,0)$; $\mathbf{P(i,j)}$: probability that a gap vector is (i,j).

Theorem 1 (1-dimensional Gap Distribution)

Let n be a positive integer, consider the distribution of gap vectors on the 1-dimensional lattice among all simple jump paths with starting point n. The distribution converges to a geometric decay as $n \to \infty$.

$$\lim_{n\to\infty} P(v) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-v-1} s_i s_{n-v-i} + 2 \cdot s_{n-v}}{g_n} = \lim_{n\to\infty} \frac{(n-v+3)2^{n-v-2}}{(n+1)\,2^{n-2}} = \frac{1}{2^v}.$$

Theorem 2 (2-dimensional Gap Vector Distribution)

Let n be a positive integer, consider the distribution of gap vectors among all simple jump paths on the 2-dimensional lattice grid with starting point (n,n). The distribution converges to a bivariate geometric decay as $n \to \infty$.

$$S(v_1,v_2) = \sum_{i=1}^{n-v_1-1}\sum_{j=1}^{n-v_2-1} \binom{i+j-2}{i-1}\binom{2n-v_1-i-v_2-j-2}{n-v_1-i-1} + 2\binom{2n-v_1-v_2-2}{n-v_1-1}.$$

A nice binomial identity

$$\sum_{k=0}^{n} \binom{x+n-k-1}{n-k}\binom{y+k-1}{k} = \binom{x+y+n-1}{n} \qquad (1)$$

miraculously allows us to simplify $S(v_1,v_2)$ to a closed form and evaluate the limit

$$\lim_{n\to\infty} P(v_1,v_2) = \lim_{n\to\infty} \frac{S(v_1,v_2)}{g_{n,n}} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{2}{n+1}\right)\frac{2}{2^{v_1+v_2}} = \frac{1}{2^{v_1+v_2}}.$$

Theorem 3 (Gap Sum Distribution)

Let n be a positive integer, the probability $P_s(k)$ that a gap sum equals k on the 2-dimensional lattice grid with starting point (n,n) converges to $(k-1)/2^k$ as $n \to \infty$.

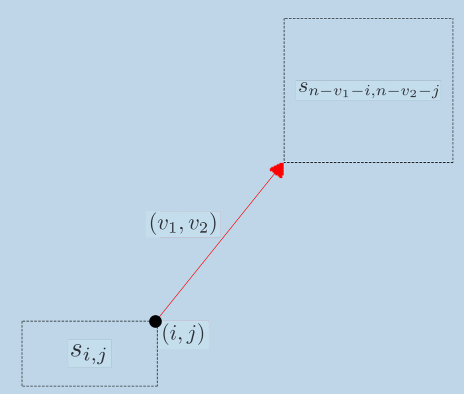

Figure 4: Illustration of proof for 2-dimensional gap vector distribution.

Future Work

We hope to generalize the results to d-dimensional lattices. The main difficulty is that combinatorial miracles like (1) generally do not have analogues in higher dimensions, thus the expression of the probability becomes much more involved.

Conj (d-dimensional Gap Vector)

As $n \to \infty$, the distribution of the gap vectors in the Zeckendorf decompositions from d-dimensional lattice grid approaches multivariate geometric decay.
We proved that

$$P(v_1,\ldots,v_d) = 2\frac{g_{n-v_1,\ldots,n-v_d}}{g_{n,\ldots,n}} + \sum_i \frac{g_{i_1,\ldots,i_d}\, g_{n-i_1-v_1,\ldots,n-i_d-v_d}}{g_{n,\ldots,n}}$$

where $g_{a_1,\ldots,a_d} = \sum_{k=1}^{\infty} k\binom{a_1-1}{k-1}\cdots\binom{a_d-1}{k-1}$. We showed the first term goes to zero as $n \to \infty$, and it remains to prove convergence for the second term.

Conj (d-dimensional Gap Sum)

As $n \to \infty$, the distribution of the gap sums in the Zeckendorf decompositions from d-dimensional lattice grid approaches geometric decay.

Generalization to Euclidean Distances

Our method can potentially be generalized to study the distribution of the Euclidean distances between summands. The analysis involves counting the number of diophantine equations that have solutions within the range of $\{1,2,\ldots,n\}$.

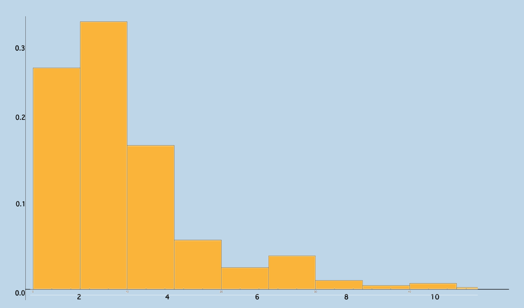

Figure 5: Distribution of Euclidean distances between summands starting at $(10,10)$.