Generalizing Zeckendorf's Theorem to Homogeneous Linear Recurrences

Clay Mizgerd
Williams College
cmm12@williams.edu

Chenyang Sun
Williams College
cs19@williams.edu

Joint work with Thomas Martinez, Advised by Steven Miller

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Past PLRS Results

Fibonacci Numbers: $F_n = F_{n-1} + F_{n-2}$.
First few: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

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Zeckendorf's Theorem

Every positive integer can be written as the sum of non-consecutive Fibonacci numbers.

Example

$84 = 55 + 21 + 8 = F_9 + F_7 + F_5$.

Definition: Positive Linear Recurrence Sequence

Definition (PLRS) from [KKMW]

A *Positive Linear Recurrence Sequence* (PLRS) is a sequence $\{H_n\}$ satisfying

$$H_n = c_1 H_{n-1} + c_2 H_{n-2} + \dots + c_L H_{n-L}$$

with non-negative integer coefficients c_i with $c_1, c_L \geq 1$ and initial values $H_1 = 1$ and $H_n = c_1 H_{n-1} + c_2 H_{n-2} + \dots + c_{n-1} H_1 + 1$ for $2 \leq n \leq L$.

PLRS Examples

Fibonacci numbers: $L = 2$, $c_1 = c_2 = 1$.
$H_1 = 1,\ H_2 = 2,\ H_3 = 3,\ H_4 = 5, \ldots$

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Fibonacci numbers: $L = 2$, $c_1 = c_2 = 1$.
$H_1 = 1$, $H_2 = 2$, $H_3 = 3$, $H_4 = 5$, ...

Powers of b: $L = 2$, $c_1 = b - 1$, $c_2 = b$.
$H_1 = 1$, $H_2 = b$, $H_3 = b^2$, $H_4 = b^3$, ...

Definition: PLRS Legal Decomposition

Definition (PLRS Legal Decomposition) from [KKMW]

Let $\{H_n\}$ be a PLRS and N a positive integer. Then,

$$N = \sum_{i=1}^{m} a_i H_{m+1-i}$$

is a **legal** decomposition if $a_1 > 0$, the other $a_i \geq 0$, and one of the following conditions hold:

- We have $m < L$ and $a_i = c_i$ for $1 \leq i \leq m$.
- There exists $s \in \{1, ..., L\}$ such that $a_1 = c_1, a_2 = c_2, ..., a_s < c_s$, and $\{b_n\}_{i=1}^{m-s}$ (with $b_i = a_{s+i}$ is either legal or empty.

PLRS Legal Examples

Example

Consider the PLRS

$$H_n = 4H_{n-1} + 3H_{n-2} + 0H_{n-3} + 3H_{n-4}.$$

Examples of NOT legal decompositions:

- $N = 5H_6$.
- $N = 4H_6 + 3H_5 + H_4$.
- $N = 4H_6 + 3H_5 + 3H_3$.

Examples of legal decompositions:

- $N = 4H_8 + 3H_7 + H_5 + 3H_3$.
- $N = H_5 + 4H_4 + H_3$.

Generalized Zeckendorf's Theorem

Generalized Zeckendorf's Theorem

Let $\{H_n\}$ be a PLRS. Then there is a unique legal decomposition for every positive integer N.

Motivating Questions

What if $c_1 = 0$?

Can we get similar results?

Two Methods

First, we will define the object we are studying (ZLRS)

Then, we explain the two methods used to understand it:

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Then, we explain the two methods used to understand it:

- Generalize the definition of legal decomposition, to include ZLRS's.
- Find a way to convert every ZLRS to a PLRS, which we know much more about.

Definition: ZLRS

Definition (ZLRS)

An s-deep *Zero Linear Recurrence Sequence* (ZLRS) is a sequence $\{G_n\}$ satisfying

$$G_n = c_1 G_{n-1} + c_2 G_{n-2} + \dots + c_s G_{n-s} + \dots + c_L G_{n-L}$$

with non-negative integer coefficients c_i with $c_s, c_L \geq 1$, $c_i = 0$ for all $1 \leq i < s$, and $L > s > 1$. Finally, let $S = \{s, \dots, L\}$ be the set of indices of positive coefficients. We need $\gcd\{S\} = 1$.

The final condition is to prevent sequences like

$$H_n = H_{n-2} + H_{n-4}.$$

Goals

Only considering 1-deep ZLRS's. So,

$$G_n = c_1 G_{n-1} + c_2 G_{n-2} + \ldots + c_L G_{n-L},$$

with $c_1 = 0$, $c_2, c_L \geq 1$, and one of the indices of the positive coefficients is odd.

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Example

“Lag-onacci" Sequence: $G_n = G_{n-2} + G_{n-3}$. First few terms: $1, 2, 4, 3, 6, 7, 9, 13, \ldots$

Main Idea

We want to generalize the PLRS legal definition to include 1-deep ZLRS's.

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For any given integer N that does not appear in the sequence $\{G_n\}$, a legal decomposition cannot use the largest integer $G_m < N$.

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For any given integer N that does not appear in the sequence $\{G_n\}$, a legal decomposition cannot use the largest integer $G_m < N$.

If N appears in the sequence as G_m, then $N = G_m$ is a legal decomposition

Example

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Consider the Lag-onaccis:
$1, 2, 4, 3, 6, 7, 9, 13, 16, \ldots$
$10 = ?$

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$10 = 7 + 3$

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Consider the Lag-onaccis:
$1, 2, 4, 3, 6, 7, 9, 13, 16, \ldots$
$10 = 7 + 3 = 6 + 4$

Loss of uniqueness..

Definition of ZLRS Legal Decomposition

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is a **legal** decomposition if $a_1 = 0, a_2 > 0$ the other $a_i \geq 0$, and one of the following conditions hold:

- We have $m < L$ and $a_i = c_i$ for $1 \leq i \leq m$.
- There exists $s \in \{2, ..., L\}$ such that $a_1 = c_1, a_2 = c_2, ..., a_s < c_s$, and $\{b_n\}_{i=1}^{m-s}$ (with $b_i = a_{s+i}$ is either legal or empty.

Initial Conditions Explained

Initial conditions are constructed so that we guarantee existence.

We want $G_1 = 1$, $G_2 = 2$, $G_3 = c_2 + 1$, $G_4 = 2c_2 + c_1 + 1$...

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We want $G_1 = 1$, $G_2 = 2$, $G_3 = c_2 + 1$, $G_4 = 2c_2 + c_1 + 1$...

What if $c_2 = 1$......?

Initial Conditions

Initial Conditions

$G_1 = 1$, $G_2 = 2$, and for $3 \leq n \leq L$,

$$G_n = \begin{cases} n & c_2 = 1 \\ c_2 G_{n-2} + c_3 G_{n-3} + \ldots + c_{n-1} G_1 + 1 & c_2 > 1 \end{cases}.$$

One exception: Lag-onaccis ($c_2 = 1$ but initial conditions are $1, 2, 4$.)

Existence

Theorem (Martinez, Mizgerd, Sun ’19)

For any 1-deep ZLRS and positive integer N, the greedy algorithm terminates in a legal decomposition.

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Proof sketch: Use strong induction!

Our ZLRS looks like

$$G_n = c_2 G_{n-2} + c_3 G_{n-3} + \ldots + c_L G_{n-L}.$$

Base cases: Show all integers up to the last initial condition has a decomposition from the greedy algorithm

Existence continued

Inductive Step: Let $G_{n-1} \leq N < G_n$. Suppose $N > G_{n-1}$. Let $m < N$ be the largest integer constructed by $c_2 G_{n-2} + c_3 G_{n-3} + ... + c_i G_{n-L}$ legally, where $c_i \leq c_L - 1$.

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For our theorem to be true, we just need $N - m$ to be expressible with the remaining terms... We need $N - m < G_{n-L}$.

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For our theorem to be true, we just need $N - m$ to be expressible with the remaining terms... We need $N - m < G_{n-L}$.

Suppose, for contradiction, $N - m \geq G_{n-L}$. Then we can increase c_i, and didn't make the largest possible m... contradiction! So we are done.

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Theorem (Martinez, Mizgerd, Sun ’19)

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with $c_2, c_3 \in \mathbb{Z}^+$, we do not have uniqueness of decompositions.

Working on generalization of this theorem to all 1-deep ZLRS. Proof based on work done in [CFHMN].

What do we mean by "Conversion"

Definition

A recurrence b_n is *derived from* a_n if the characteristic polynomial of a_n divides that of b_n.

If the initial values of b_n satisfy a_n, then the two recurrences produce the same sequence.

Given a ZLRS a_n, we seek to construct a PLRS of minimal order derived from a_n.

Tree Algorithm

$$a_n = a_{n-\alpha} + a_{n-\beta}$$

a_n

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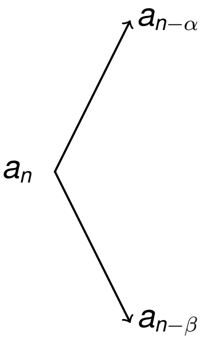

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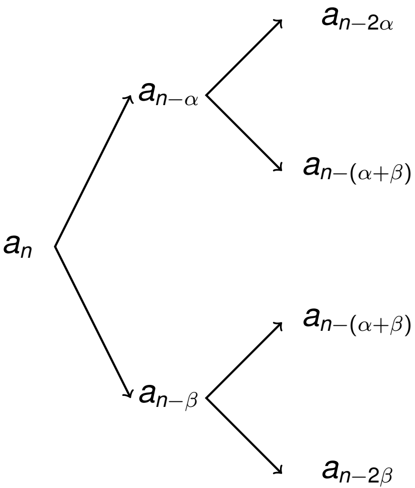

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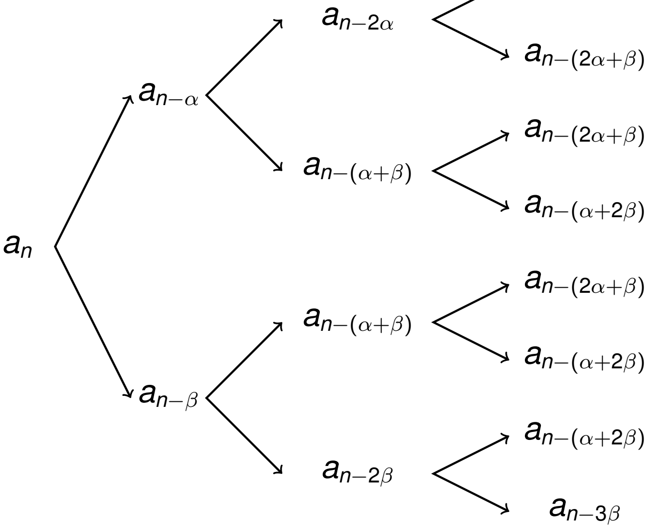

Example

$$a_n = a_{n-2} + a_{n-3}$$

a_n a_{n-1}

Example

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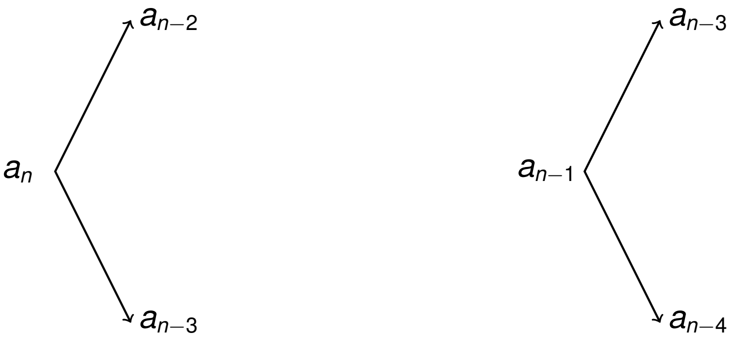

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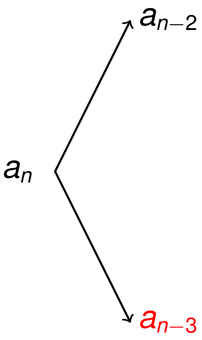
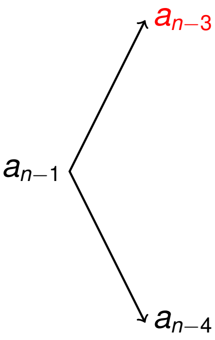
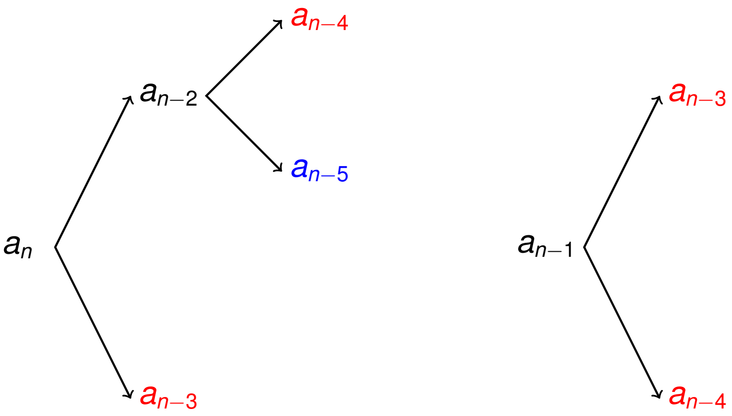




Example

$$a_n = a_{n-2} + a_{n-3}$$



Example

$$a_n = a_{n-2} + a_{n-3}$$
$$a_n = a_{n-1} + a_{n-5}$$

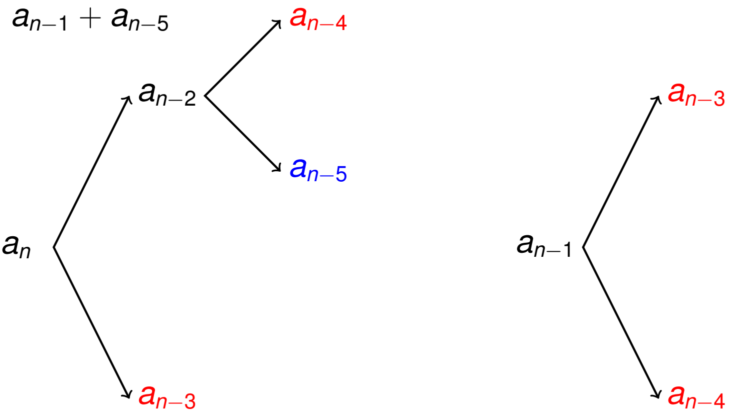


Table Algorithm

$$a_n = a_{n-2} + a_{n-3}$$

a_n	1	0
a_{n-1}	0	1
a_{n-2}	0	0
a_{n-3}	0	0
a_{n-4}	0	0
a_{n-5}	0	0
a_{n-6}	0	0

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a_{n-5}	0	0
a_{n-6}	0	0

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a_{n-6}	0	0

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a_{n-1}	0	0
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a_{n-4}	1	1
a_{n-5}	1	0
a_{n-6}	0	0

General Algorithm

$$a_n = a_{n-3} + a_{n-7}$$

a_{n-74}	11802
a_{n-75}	14422
a_{n-76}	16375
a_{n-77}	6515
a_{n-78}	7685
a_{n-79}	9369
a_{n-80}	10345
a_{n-81}	0

General Algorithm

$$a_n = a_{n-3} + a_{n-7}$$

a_{n-74}	11802	w_{74}
a_{n-75}	14422	w_{75}
a_{n-76}	16375	w_{76}
a_{n-77}	6515	
a_{n-78}	7685	
a_{n-79}	9369	
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a_{n-75}	14422	w_{75}
a_{n-76}	16375	w_{76}
a_{n-77}	6515	$w_{77} - w_{74}$
a_{n-78}	7685	$w_{78} - w_{75}$
a_{n-79}	9369	$w_{79} - w_{76}$
a_{n-80}	10345	$w_{80} - w_{77}$
a_{n-81}	0	

General Algorithm

$$a_n = a_{n-3} + a_{n-7}$$

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a_{n-75}	14422	w_{75}
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a_{n-77}	6515	$w_{77} - w_{74}$
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$$w_n = w_{n-3} + w_{n-7}$$
$$w_0 = 1$$
$$w_i = 0 \text{ if } i < 0$$

General Algorithm

$$a_n = a_{n-\alpha} + a_{n-\beta}$$

$a_{n-(t+1)}$	w_{t+1}	$w_{(t+1)-1}$
$a_{n-(t+2)}$	w_{t+2}	$w_{(t+2)-1}$
$\dots$	$\dots$	$\dots$
$a_{n-(t+\alpha-1)}$	$w_{t+\alpha-1}$	$w_{(t+\alpha-1)-1}$
$a_{n-(t+\alpha)}$	$w_{t+\alpha-\beta}$	$w_{(t+\alpha-\beta)-1}$
$a_{n-(t+\alpha+1)}$	$w_{t+\alpha+1-\beta}$	$w_{(t+\alpha+1-\beta)-1}$
$\dots$	$\dots$	$\dots$
$a_{n-(t+\beta)}$	$w_{t+\beta-\beta}$	$w_{(t+\beta-\beta)-1}$

$$w_n = w_{n-\alpha} + w_{n-\beta}$$
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Final Proof

Lemma

Any recurrence relation with relatively prime indices and irreducible characteristic polynomial is eventually either indefinitely zero, monotonically increasing, or monotonically decreasing.

$w_{t+i} > w_{t+i-1}$ for sufficiently large t.

Theorem (Martinez, Mizgerd, Sun '19)

Any (two-term) ZLR can be converted into a derived PLR.

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