# Juggling Coefficients in Complete Recurrent Sequences

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# Introduction

# Motivation

- **Positive linear recurrence sequences** (PLRS) generalize the Fibonacci numbers in Zeckendorf's theorem.
- **Complete** sequences can be used as numeration systems to express integers.

**Research Question**

How can we determine whether a PLRS is complete based on the coefficients in its defining recurrence relation?

**Definition**

A sequence $\{H_i\}_{i\geq 1}$ of positive integers is a **Positive Linear Recurrence Sequence (PLRS)** if:

- (Recurrence relation) There are non-negative integers $L, c_1, \ldots, c_L$ such that
$$H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L}$$
with $L, c_1, c_L$ positive.
- (Initial conditions) $H_1 = 1$, and for $1 \leq n \leq L$,
$$H_{n+1} = c_1 H_n + \cdots + c_n H_1 + 1$$

# Positive Linear Recurrence Sequences

- Write $[c_1, \dots, c_L]$ for $H_{n+1} = c_1 H_n + \cdots + c_L H_{n-L+1}$.
- Fibonacci numbers: $[1, 1]$. Initial conditions $F_1 = 1,\ F_2 = 2$.
- (Lucas and Pell numbers are not PLRS, due to initial conditions).

# Introduction to Completeness

**Definition**

A sequence $\{H_i\}_{i \geq 1}$ is **complete** if every positive integer is a sum of its terms, using each term at most once.

- The sequence $[1, 3]$ is *not* complete. Its terms are $\{1, 2, 5, 11, \dots\}$; you cannot get 4 or 9.
- The Fibonacci sequence is complete (follows from Zeckendorf's Theorem).

# The Doubling Sequence $H_{n+1} = 2H_n$

The PLRS [2] has terms $H_n = 2^{n-1}$, i.e., $\{1, 2, 4, 8, \ldots\}$, and is complete (every integer has a binary representation).

**Theorem (Brown)**

*The complete sequence with maximal terms is $H_n = 2^{n-1}$.*

Any PLRS of the form $[1, \ldots, 1, 2]$ has the same terms as [2], i.e., $H_n = 2^{n-1}$.

# Brown's Criterion

**Theorem (Brown)**

*A nondecreasing sequence* $\{H_i\}_{i\geq 1}$ *is complete if and only if* $H_1 = 1$ *and for every* $n \geq 1$*,*

$$H_{n+1} \;\leq\; 1 + \sum_{i=1}^{n} H_i.$$

**Definition**

The ***n*-th Brown's Gap** of a sequence $\{H_i\}_{i\geq 1}$ is

$$B_{H,n} \;:=\; 1 + \left(\sum_{i=1}^{n-1} H_i\right) - H_n.$$

# Modifying Sequences

# Example for $L = 6$

**Example**

$[1, 0, 0, 0, 0, N]$ is complete if and only if $N \leq 11$.

**Question**

Does there exist a complete PLRS of length $L = 6$, i.e., with coefficients $[c_1, \ldots, c_5, N]$, with $N > 11$?

# Example for $L = 6$

- $[1, 0, 0, 0, 0, N]$ is complete for $N \leq 11$.
- $[1, 1, 0, 0, 0, N]$ is complete for $N \leq 11$.
- $[1, 0, 1, 0, 0, N]$ is complete for $N \leq 12$.
- $[1, 0, 0, 1, 0, N]$ is complete for $N \leq 11$.
- $[1, 0, 0, 0, 1, N]$ is complete for $N \leq 10$.

Why is $[1, 0, 1, 0, 0, 12]$ complete, but $[1, 0, 0, 0, 0, 12]$ is not complete?

# Example for $L = 6$

Why is $[1, 0, 1, 0, 0, 12]$ complete, but $[1, 0, 0, 0, 0, 12]$ is not complete?

- $[1, 0, 0, 0, 0, 12]$ has terms $\{1, 2, 3, 4, 5, 6, 18, 42, \dots\}$ and so computing $1 + \sum_{i=1}^{n} H_i$ we see $\{2, 4, 7, 11, 16, 22, 40, \dots\}$
- $[1, 0, 1, 0, 0, 12]$ has terms $\{1, 2, 3, 5, 8, 12, 29, 61, \dots\}$ and so computing $1 + \sum_{i=1}^{n} H_i$ we see $\{2, 4, 7, 12, 20, 32, 61, \dots\}$
- $[1, 1, 1, 0, 0, 12]$ has terms $\{1, 2, 4, 8, 15, 28, 63, \dots\}$ and so computing $1 + \sum_{i=1}^{n} H_i$ we see $\{2, 4, 8, 16, 31, 59, \dots\}$

# Modifying Coefficients of a PLRS

What modifications to the coefficients preserve completeness or incompleteness?

**Theorem (SMALL 2020)**

*If $[c_1, \dots, c_L]$ is any incomplete sequence, then the sequence $[c_1, \dots, c_{L-1} + c_L]$ is also incomplete.*

**Theorem (SMALL 2020)**

*If a sequence $[c_1, \dots, c_{L-1}, c_L]$ is complete, then so is $[c_1, \dots, c_{L-1}, d_L]$ for any $1 \leq d_L \leq c_L$.*

*Remark. Not true for $c_i$ in an arbitrary position.*

We discuss bounds for the last coefficient.

# Families of Sequences

# Analyzing Families of Sequences

**Theorem (SMALL 2020)**

- $[1, \underbrace{0, \dots, 0}_{k}, N]$, *is complete if and only if*

$$N \leq \left\lfloor \frac{(k+2)(k+3)}{4} + \frac{1}{2} \right\rfloor.$$

- $[1, 1, \underbrace{0, \dots, 0}_{k}, N]$, *is complete if and only if*

$$N \leq \left\lfloor \frac{F_{k+6} - (k+5)}{4} \right\rfloor,$$

*where $F_k$ is the $k$th Fibonacci number.*

# Proof Sketch

**Theorem**

$[1, 0, \ldots, 0, N]$, *with* $k$ *zeros, is complete if and only if* $N \leq \left\lfloor \frac{(k+2)(k+3)}{4} + \frac{1}{2} \right\rfloor$.

*Partial Proof.* We sketch that if $N_{\max} = \left\lfloor \frac{(k+2)(k+3)}{4} + \frac{1}{2} \right\rfloor$, then the sequence is complete.

With the recurrence relation and Brown's criterion,

$$\begin{aligned} H_{n+1} &= H_n + N_{\max} H_{n-k-1} \\ &\leq H_n + (N_{\max} - 1) H_{n-k-1} + H_{n-k-2} + \cdots + H_1 + 1 \end{aligned}$$

By induction, $(N_{\max} - 1) H_{n-k-1} \leq H_{n-1} + \cdots + H_{n-k-1}$, so

$$\leq H_n + \cdots + H_1 + 1.$$

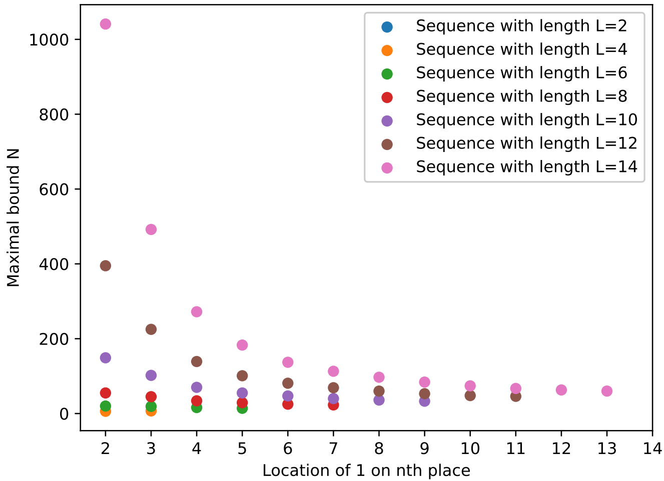


**Figure 1:** $[1, \underbrace{0, \ldots, 0}_{k}, 1, \underbrace{0, \ldots, 0}_{g}, N]$ with location of middle one varying, where each color represents a fixed length $L$.

# Theorem on Switching Ones

**Theorem (SMALL 2020)**

*Let $L \geq 6$ fixed and* $\{H_n\} = [1, \underbrace{0, \dots, 0}_{L-g-3}, 1, \underbrace{0, \dots, 0}_{g}, M]$, $0 < g \leq L-3$. *If $M$ is maximal such that $\{H_n\}$ is complete, and $N$ is maximal such that $[1, 0, \dots, 0, N]$ is complete,* $M \geq N$.

*In particular,*

- $[1, 0, \dots, 0, 0, 1, M]$ *is complete if and only if* $M \leq N - 1$
- $[1, 0, \dots, 0, 1, 0, M]$ *is complete if and only if* $M \leq N$.

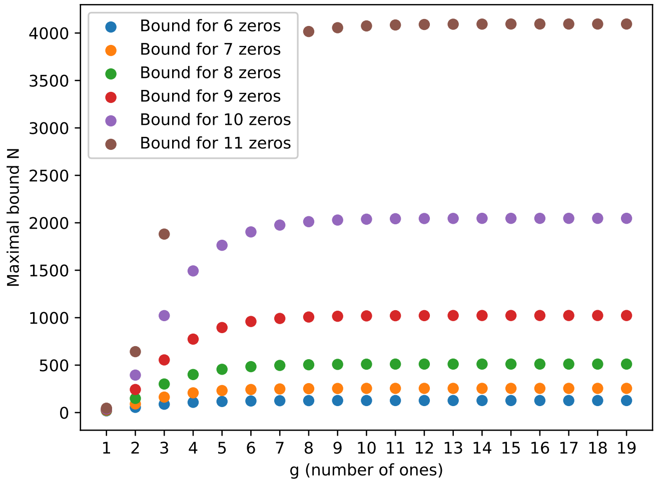


**Figure 2:** $[\underbrace{1,\ldots,1}_{g},\underbrace{0,\ldots,0}_{k},N]$ with $k$ and $g$ varying, where each color represents a fixed $k$.

# Sequences of Initial Ones

**Theorem (SMALL 2020)**

*If a sequence* $[\underbrace{1,\ldots,1}_{g},\underbrace{0,\ldots,0}_{k},N]$ *is complete with* $g \geq 3$,

*then*

$$N \leq \frac{1}{2}\left(1+\sum_{i=1}^{k+1} F_i^{(g)}+\sum_{i=1}^{k+1-g} F_i^{(g)}+\cdots+\sum_{i=1}^{(k+1) \bmod g} F_i^{(g)}\right)$$

*where* $F_i^{(g)}$ *is the* $g$*-bonacci sequence,* $[\underbrace{1,\ldots,1}_{g}]$.

# Sequences of Initial Ones

**Conjecture (SMALL 2020)**

If a sequence $[\underbrace{1, \ldots, 1}_{g}, \underbrace{0, \ldots, 0}_{k}, N]$ is complete, then so is $[\underbrace{1, \ldots, 1}_{g+j}, \underbrace{0, \ldots, 0}_{k}, N]$ for any positive integer $j$.

**Theorem (SMALL 2020)**

*Consider* $[\underbrace{1, \ldots, 1}_{g}, \underbrace{0, \ldots, 0}_{k}, N]$.

- *For* $g \geq k + \lceil \log_2 k \rceil$, *the bound on* $N$ *is* $N \leq 2^{k+1} - 1$
- *For* $k \leq g < k + \lceil \log_2 k \rceil$, *the bound on* $N$ *is* $N \leq 2^{k+1} - \left\lceil \frac{k}{2^{g-k}} \right\rceil$

# The $2L - 1$ conjecture

# The $2L-1$ conjecture

Can we bound where a sequence must fail Brown's Criterion?
We think so!

**Conjecture (SMALL 2020)**

If a PLRS $H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L}$ incomplete, then it fails Brown's criterion before the $2L$-th term.

The closest we've gotten:

**Theorem (SMALL 2020)**

*The PLRS $\{H_i\}_{i\geq 1}$ generated by $[c_1, \ldots, c_L]$ is complete if*

$$\begin{cases} B_{H,n} \geq 0, \ 1 \leq n < L \\ B_{H,n} > 0, \ L \leq n \leq 2L-1 \end{cases}$$

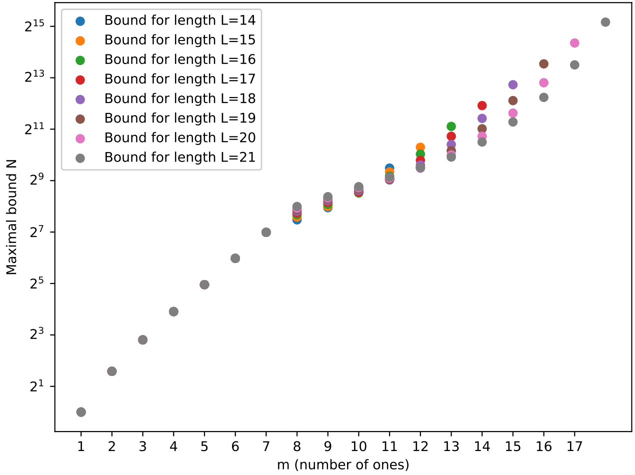


**Figure 3:** $[1, \underbrace{0, \ldots, 0}_{k}, \underbrace{1, \ldots, 1}_{m}, N]$ with number of ones $(m)$ varying, depending on $L$.

## Conditional result on Adding Ones

If the $2L-1$ conjecture holds, we have the following:

**Theorem (SMALL 2020)**

For a fixed length $L$, the sequence $[1, \underbrace{0, \ldots, 0}_{k}, \underbrace{1, \ldots, 1}_{m}, N]$ with $m$ ones has a lower bound on $N$ than the sequence $[1, \underbrace{0, \ldots, 0}_{k-1}, \underbrace{1, \ldots, 1}_{m+1}, N]$.

In particular, if $m < \frac{L}{2}$, the bound is precisely

$$N \leq \left\lfloor \frac{(L-m)(L+m+1)}{4} + \frac{1}{48}m(m+1)(m+2)(m+3) + \frac{1-2m}{2} \right\rfloor .$$

# Binet's Formula and Generalizations

# Characteristic Polynomials

**Definition**

For a PLRS $\{H_n\}$ defined by $[c_1, \ldots, c_L]$, define the characteristic polynomial

$$p(x) = x^L - \sum_{i=1}^{L} c_i x^{L-i}.$$

- By Descartes's Rule of Signs, $p(x)$ there is one positive real root, the **principal root**.
- The principal root is always the largest: for any root $z \in \mathbb{C}$, $|z| < r$.

# Generalized Binet's Formula

**Theorem (Generalized Binet's Formula)**

*If $r_1, \dots, r_k$ are the roots of the polynomial of a linear recurrence $\{H_n\}$ with multiplicities $m_1, \dots, m_k$, there are polynomials $q_1, \dots, q_k$ with $\deg(q_i) \leq m_i - 1$ such that*

$$H_n = q_1(n) r_1^n + \ldots + q_k(n) r_k^n.$$

- If $\{H_n\}$ is a PLRS, let $r_1$ be the principal root; since $m_1 = 1$ and for all $i, r_1 > |r_i|$, then $H_n = \Theta(r_1^n)$.
- Complete sequences should grow "slowly". Can we relate the size of $r_1$ to completeness?

# Bounding the Principal Root

# First Bounds on $r_1$

Recall $p(x) = x^L - \sum_{i=1}^{L} c_i x^{L-i}$.

As $c_L \geq 1$, we know $r_1 > 1$. ($c_L = \prod r_i^{m_i}$, and $r_1$ is the biggest root by magnitude).

**Lemma (SMALL 2020)**

*If $H_n$ is a complete PLRS and $r_1$ is its principal root, then $r_1 \leq 2$.*

**Proof.**

Otherwise, as $H_n = \Theta\left(r_1^n\right)$, for large $n$ our terms would exceed the maximal sequence $\{2^{n-1}\}$. □

Note: there are incomplete sequences with principal roots $r \leq 2$.

# Is 2 a Useful Bound?

- We can find complete sequences with roots of sizes arbitrarily close to 2. (Sequences of the form $[\underbrace{1, \ldots, 1}_{L}]$.)
- Checking $r_1 \leq 2$ is a fast method to eliminate candidates for completeness.
- $p(x) = x^L - \sum_{i=1}^{L} c_i x^{L-i}$ has one positive real root, and $p(x) > 0$ for large $x$, so $r_1 \leq 2$ if and only if $p(2) \geq 0$. This is much faster than checking Brown's Criterion!

## Lower Bound

**Lemma (SMALL 2020)**

*For any $L$, there exists a second bound $B_L$ such that if a sequence $[c_1, \ldots, c_L]$ is incomplete, then $r_1 \geq B_L$.*

**Proof.**

- There are finitely many sequences $[c_1, \ldots, c_L]$ with $p(2) = 2^L - \sum_{i=1}^{L} c_i 2^{L-i} \geq 0$.
- Hence finitely many incomplete sequences with $r_1 \leq 2$, so just find the minimum root - $B_L$.

$\square$

We now aim to determine the precise values of $B_L$.

# The Minimal Incomplete Sequence

**Theorem (SMALL 2020)**

$[1, \underbrace{0, \ldots, 0}_{L-2}, N]$, *is complete if and only if*

$$N \leq \left\lceil \frac{L(L+1)}{4} \right\rceil .$$

**Conjecture (SMALL 2020)**

*For any $L$, the incomplete sequence of length $L$ with smallest principal root is $[1, 0, \ldots, 0, \left\lceil \frac{L(L+1)}{4} \right\rceil + 1]$.*

- Let $\lambda_L$ the principal root of $[1, 0, \ldots, 0, \left\lceil \frac{L(L+1)}{4} \right\rceil + 1]$. This is saying $\lambda_L = B_L$, for all $L$.

# Denseness of Incomplete Roots

**Theorem (SMALL 2020)**

*For any $L \in \mathbb{Z}^+$, let $R_L$ be the set of roots of all incomplete PLRS of length L. Then, for any $\varepsilon > 0$, there exists an M such that for all $L > M$, for any $\varepsilon$-ball $B_\varepsilon \subset [1, 2]$, $B_\varepsilon \cap R_L \neq \varnothing$.*

**Corollary**

*The set $R = \bigcup_{L=1}^{\infty} R_L$ of all principal roots of incomplete sequences is dense in $[1, 2]$.*

# Bibliography


- Olivia Beckwith, Amanda Bower, Louis Gaudet, Rachel Insoft, Shiyu Li, Steven J. Miller, and Philip Tosteson. The Average Gap Distribution for Generalized Zeckendorf Decompositions, Dec 2012.
- J. L. Brown. Note on complete sequences of integers. *The American Mathematical Monthly*, 68(6):557, 1961.
- Aviezri S. Fraenkel. The use and usefulness of numeration systems. *Information and Computation*, 81(1):46-61, 1989.
- Thomas C. Martinez, Steven J. Miller, Clay Mizgerd, and Chenyang Sun. Generalizing Zeckendorf's Theorem to Homogeneous Linear Recurrences, 2020

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- Thank you. Any questions?

# Appendix

# Proof of Denseness Theorem

We use that the $\lambda_L$ roots are decreasing, and $\lim_{L\to\infty} \lambda_L = 1$.

**Proof.**

Consider the following incomplete sequences:

$$[1, 0, \ldots, 0, \left\lceil \frac{L(L+1)}{2} \right\rceil + 1],\ [1, 0, \ldots, 0, \left\lceil \frac{L(L+1)}{2} \right\rceil + 2],\ \ldots, [1, 0, \ldots, 0, 2^L]$$

- The root of the first sequence approaches 1.
- Roots of consecutive sequence increase at a decreasing rate.
- Root of the last sequence exceeds 2.
- Thus for $\lambda_L < 1 + \varepsilon$, roots are going up by at most $\varepsilon$.

□