Tinkering with Lattices: A New Take on the Erdős Distance Problem

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Erdős distinct distances problem

Question [Erdős, 1946]

Given n points in a plane, what is the minimum number of distinct distances $f(n)$ that they determine?

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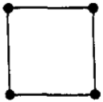
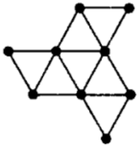
Some Examples:

3 points; 1 distance



4 points; 2 distances



9 points; 4 distances

First Estimates

Theorem (Erdős, 1946)

Let $[P_n]$ *be the class of subsets of the plane with n points, and let* $f(n)$ *be the minimum number of distinct distances determined by an element* $P_n \in [P_n]$*. Then,*

$$(n - 3/4)^{1/2} - 1/2 \ \leq \ f(n) \ \leq \ cn/\sqrt{\log n}.$$

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Upper Bound: Upper bound for distinct distances of the $\sqrt{n} \times \sqrt{n}$ integer lattice.

Lower Bound (the hard part): Work with the convex hull of an arbitrary point set P_n.

Erdős Distinct Distances Problem: Bounds

Upper bounds (unimproved since Erdős!):

- $\Delta(n) = O(\frac{n}{\sqrt{\log n}})$ (Erdős, 1946)

Lower bounds:

- $\Delta(n) = \Omega(n^{1/2})$ (Erdős, 1946)
- $\Delta(n) = \Omega(n^{4/5}/\log n)$ (Chung, Szemeredi Trotter, 1992)
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A set with $O(\frac{n}{\sqrt{\log n}})$ distinct distances is *near-optimal*. The integer lattice is a near-optimal set.

Lattice Distance Distribution

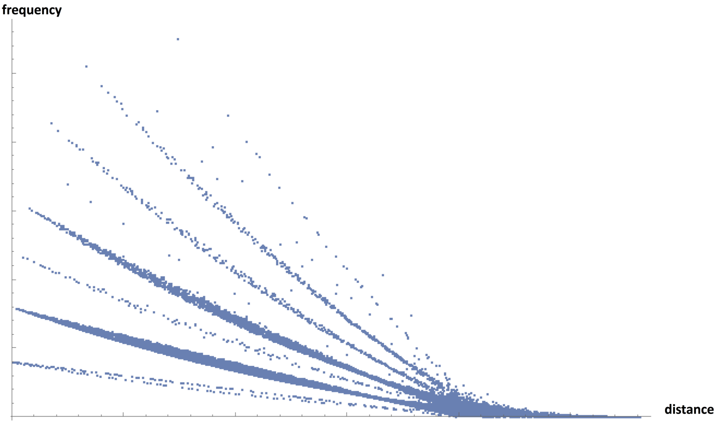
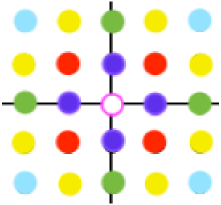


Figure: Distance distribution for 200×200 integer lattice

Repeating Distances



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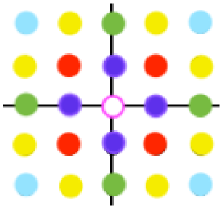
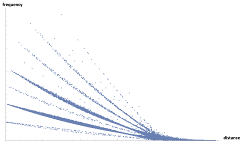


How often do distances on the integer lattice repeat?

- 4 points at a distance 1 from the origin.
- 4 points at a distance $\sqrt{2} = \sqrt{1^2 + 1^2}$ from the origin.
- 8 points at a distance $\sqrt{5} = \sqrt{2^2 + 1^2} = \sqrt{1^2 + 2^2}$.

Calculating Distance Frequency



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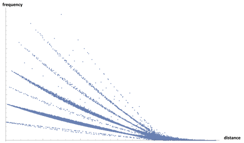

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- Find all the decompositions of d into $d = a^2 + b^2$, where $N - 1 \geq a \geq b \geq 0$.

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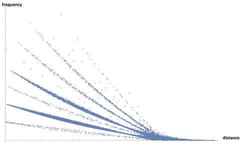

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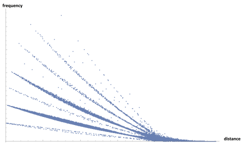

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- If $b = 0$ or $a = b$, then the frequency of that particular decomposition is $2(N - a)(N - b)$. If $a > b$ then the frequency of that particular decomposition is $4(N - a)(N - b)$.

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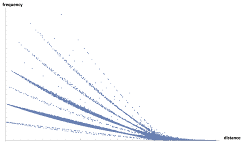

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- Add all the frequencies together.

More Facts About the Distance Distribution

Theorem (Fermat)

Suppose d has prime factorization $d = 2^f p_1^{g_1} \cdots p_m^{g_m} q_1^{h_1} \cdots q_n^{h_n}$, where $p_i \equiv 1 \pmod 4$, $q_i \equiv 3 \pmod 4$. Then there exist $r(d)$ ordered pairs $(a, b) \in \mathbb{Z}^2$ with $a^2 + b^2 = d$, where

$$r(d) = \begin{cases} 4\,(g_1+1)\cdots(g_m+1) & h_i \text{ is even for all } i, \\ 0 & \text{else.} \end{cases}$$

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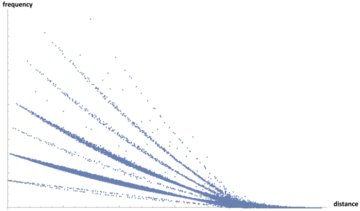
- The number of integers in the set $\{1, \ldots, 2n\}$ which can be written as the sum of two squares is of order $\frac{cn}{\sqrt{\log n}}$. (Source of Erdos's Upper Bound!)

What is the most common distance on the lattice?

- The first (left-most) distance on each curve has the highest frequency on that curve.



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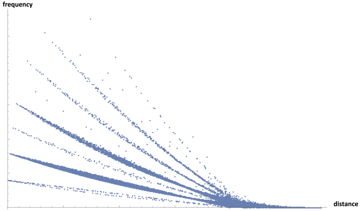
- The first (left-most) distance on each curve has the highest frequency on that curve.
- Define n_k as the least positive integer such that there are k ordered pairs (a, b) with $a^2 + b^2 = n_k$, so that $\sqrt{n_k}$ is the first distance on the k-th curve. Then the sequence $n_1, n_2, \ldots$ will be a list of potential candidates for the most common distance on the lattice!



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Lemma (SMALL 2020)

Let $k = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_n^{\alpha_n}$ *be arbitrary, where* $q_1 > q_2 > \ldots > q_n$*, and let* $5 = p_1 < p_2 < \cdots$ *be the primes* $\equiv 1 \pmod 4$*, in increasing order. Then,*

$$n_k = \Big(\underbrace{p_1 \cdots p_{\alpha_1}}_{\alpha_1 \text{ primes}}\Big)^{q_1 - 1} \Big(\underbrace{p_{\alpha_1+1} \cdots p_{\alpha_1+\alpha_2}}_{\alpha_2 \text{ primes}}\Big)^{q_2 - 1} \cdots \Big(\underbrace{p_{\alpha_1+\cdots+\alpha_{n-1}+1} \cdots p_{\alpha_1+\cdots+p_{\alpha_1+\cdots+\alpha_n}}}_{\alpha_n \text{ primes}}\Big)^{q_n - 1}$$

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- Adapting previous asymptotic results on the product of the first k primes,
$$n_k \approx e^{\frac{1}{2}(1+c)\log_2 2k \log \log_2 2k}.$$
We arrive at the following upper bound for the frequency of $\sqrt{n_k}$:
$$2kN\left(N - e^{\frac{1}{4}(1+c)\log_2 2k \log \log_2 2k}\right).$$

Error introduction

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Basically, we are trying to solve the Erdős distance problem on subsets of the lattice.

Calculating error

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- The $N \times N$ lattice has $\frac{N^2(N^2-1)}{2} \approx \frac{N^4}{2}$ total distances. A subset with p points has $\frac{p(p-1)}{2} \approx \frac{p^2}{2}$ total distances.

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- So we scale the distance distribution of the subset up by $\frac{N^4}{p^2}$.
- Then, for each unique distance we find the absolute difference between the scaled subset frequency and the lattice frequency.
- We then average these difference to find the error.

Configurations

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Figure: $p = 4$

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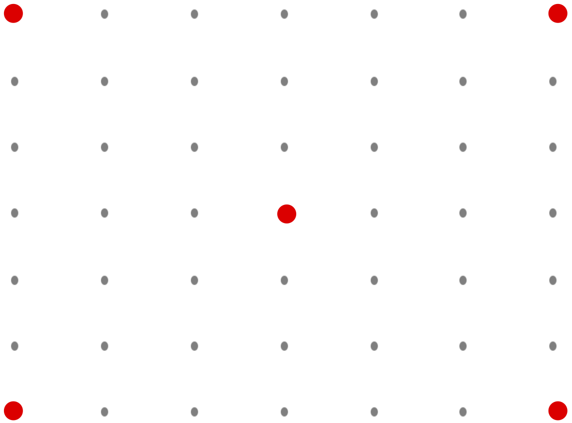

Figure: $p = 5$

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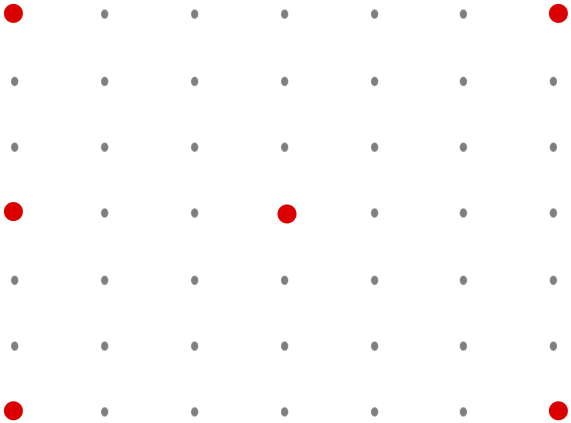

Figure: $p = 6$

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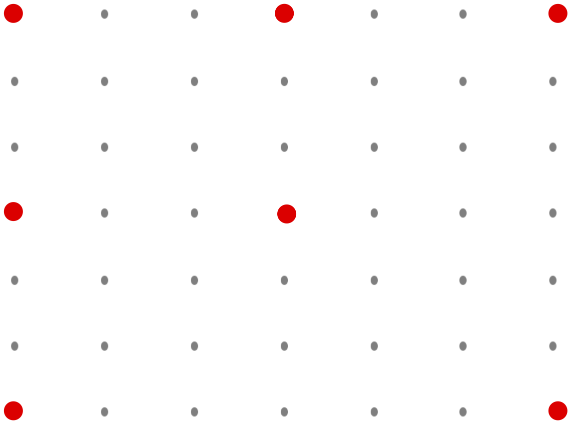

Figure: $p = 7$

Configurations

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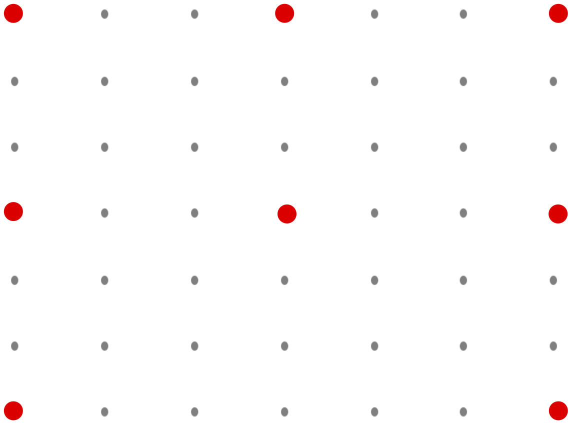

Figure: $p = 8$

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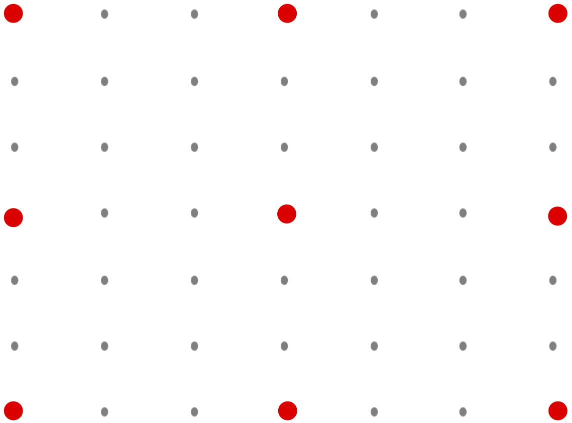

Figure: $p = 9$

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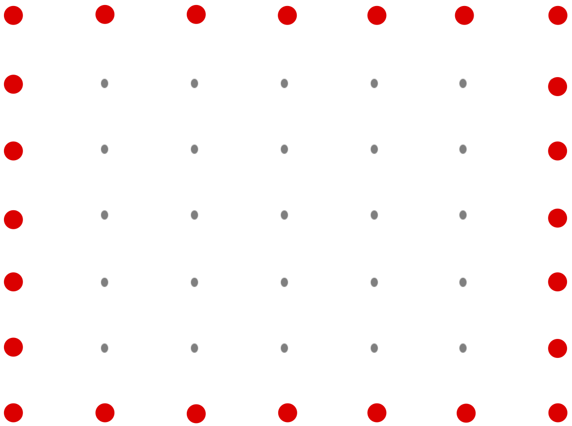

Figure: $p = 4(N-1)$

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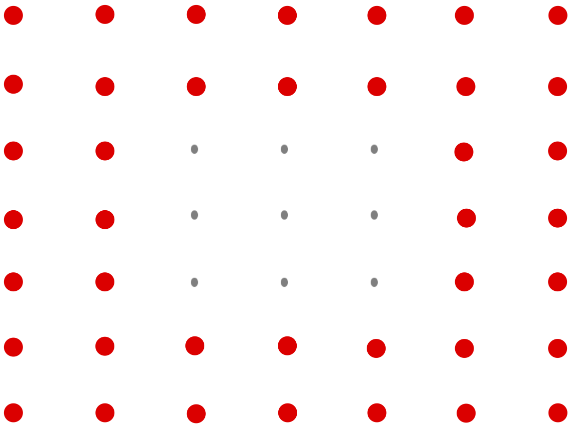

Figure: $p = 4(N-1) + 4(N-3)$

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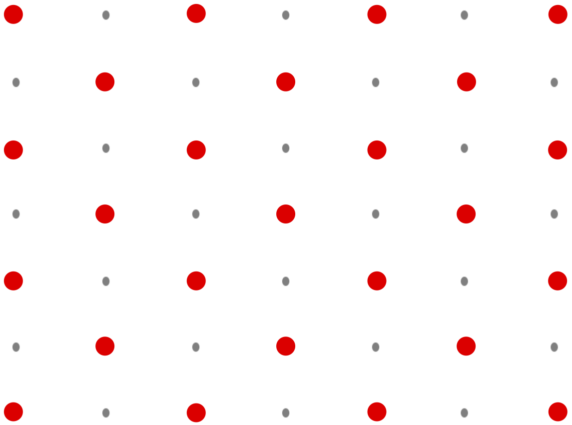

Figure: $p = \left\lceil \frac{N^2}{2} \right\rceil$

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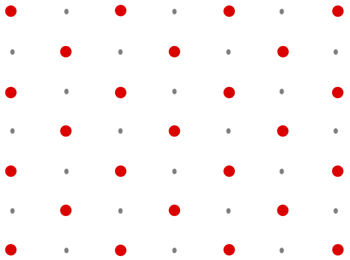
Ex: for $p = \left\lceil \frac{N^2}{2} \right\rceil$ we have a checkerboard lattice.



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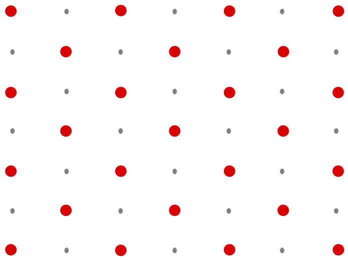

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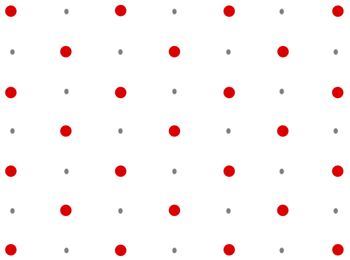

We simplify by looking at $\sqrt{a^2 + b^2}$ instead of $\sqrt{d}$.

$\sqrt{a^2 + b^2}$ only appears if a and b are both even or both odd.

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The error is:

$$\frac{4}{N+2}\left[\frac{3}{4}\left(4\left(\frac{N(5N-1)}{6}\right)-\frac{N(5N-1)}{6}\right)+\frac{1}{4}\left(\frac{N(5N-1)}{6}\right)\right]+$$

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$$=2N^2-\frac{25N}{6}-\frac{121}{21(N+2)}+\frac{188}{21(3N-1)}+\frac{71}{6}$$

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$$\frac{4}{N+2}\left[\frac{3}{4}\left(4\left(\frac{N(5N-1)}{6}\right)-\frac{N(5N-1)}{6}\right)+\frac{1}{4}\left(\frac{N(5N-1)}{6}\right)\right]+$$

$$\frac{N-2}{N+2}\left[\frac{1}{2}\left(4\left(\frac{N(3N-1)}{3}\right)-\frac{N(3N-1)}{3}\right)+\frac{1}{2}\frac{N(3N-1)}{3}\right]$$

$$=2N^2-\frac{25N}{6}-\frac{121}{21(N+2)}+\frac{188}{21(3N-1)}+\frac{71}{6}$$

Lower Bounds

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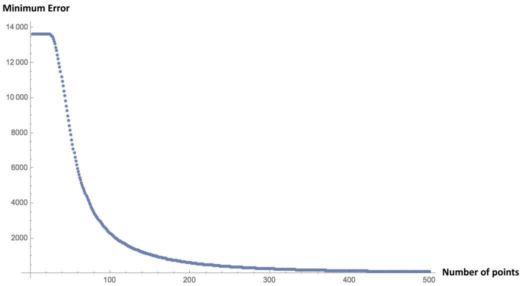
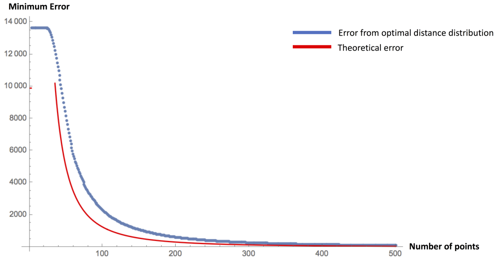


Figure: data for $N = 100$

Calculating Lower Bound

$$\text{Error} \geq \begin{cases} \frac{N^3}{N+2} + \frac{N^2}{N+2} - \frac{10N}{3(N+2)} & \text{if } p \leq \frac{\log_5(N)}{5}\left(11 - 2\sqrt{10}\right), \\ \frac{N^4}{8p^2} & \text{if } p \text{ sufficiently large.} \end{cases}$$



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$p \leq \log_5(N)(11 - 2\sqrt{10})/5$ ensures $N^4/p^2 > 2F$.

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- Some intuition: the average error should be around $\frac{p^2}{4N^4}$
- *However*, for small p, many original frequencies are very close to 0, so average is smaller than $\frac{N^4}{4p^2}$

Further work

- Characterizing sets of maximum error.
- Characterizing sets of minimum error.
- Extending results to other lattice structures.

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Questions?

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