Extending support in calculating the 1-level density of low lying zeros of families of L-functions

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Overview

A surprising connection

The zeroes of L-functions near the central point and the energy levels of heavy nuclei can both be modeled by eigenvalues of random matrices.

The limiting distribution of eigenvalues of random matrices are reasonable to calculate, while the distribution of zeroes of L-functions are elusive.

Definition

Let $H_k^*(N)$ be the family of holomorphic cuspidal newforms of weight k and level N.

Definition (L-function associated to $f \in H_k^*$)

$$L(s,f) \ := \ \sum_{n=1}^{\infty} \lambda_f(n) n^{-s},$$

where $\lambda_f(n) := a_f(n) n^{-(k-1)/2}$, for $a_f(n)$ the n^{th} Fourier coefficient.

GL_2 automorphic L-functions

The completed L-function $\Lambda(s,f)$ of $L(s,f)$ satisfies the following functional equation:

$$\Lambda(s,f) \;=\; \left(\frac{\sqrt{N}}{2\pi}\right)^{s}\Gamma\left(s+\frac{k-1}{2}\right)L(s,f) \;=\; \varepsilon_f\Lambda(1-s,f).$$

Definition

Let $H_k^{\pm}(N)$ be the set of $f \in H_k^*(N)$ with $\varepsilon_f = \pm 1$.

1-level Density

Definition (Test function)

$\phi : \mathbb{R}^n \to [0, \infty)$ is a test function if it is even, Schwartz, $\hat{\phi}$ has compact support, and $\phi(0) > 0$.

Definition (1-level density)

Let $\rho_f = 1/2 + i\gamma_f$ be a non-trivial zero of $L(s, f)$. The **1-level density** of the $L(s, f)$ is

$$D(f; \phi) := \sum_{\gamma_f} \phi\left(\frac{\log R}{2\pi}\gamma_f\right),$$

where ϕ is a test function, R is the analytic conductor, and the sum is over normalized imaginary parts γ_f of zeros of the L-function associated with f.

Averaging over the family of $L(s, f)$

For a family $H_k^{\pm}(N)$ of cuspidal newforms weight k and squarefree level N, the first moment of the 1-level density is

$$\lim_{N\to\infty} \langle D(f;\phi)\rangle_{\pm} := \lim_{N\to\infty} \frac{1}{|H_k^{\pm}(N)|} \sum_{f\in H_k^{\pm}(N)} D(f;\phi).$$

To study n^{th} centered moments of $D(f;\phi)$ we explicitly compute

$$\lim_{N\to\infty} \langle (D(f;\phi) - \langle D(f;\phi)\rangle_{\pm})^n)\rangle_{\pm}$$

for $\hat{\phi}$ of large support.

Scaling limits of eigenvalue distributions

We also study analogous random matrix theory quantities. For U a unitary $M \times M$ matrix with eigenvalues $e^{i\theta_n}$, define

Definition

For a Schwartz function ϕ on the real line, define

$$Z_\phi(U) \ := \ \sum_{n=1}^{M} \sum_{j=-\infty}^{\infty} \phi\left(\frac{M}{2\pi}(\theta_n + 2\pi j)\right).$$

Averaging over the family of matrices

For the analogous family of special orthogonal matrices, the n^{th} moment of $Z_\phi(U)$ averaged with Haar measure over SO(odd) (or respectively SO(even)) is

$$\lim_{\substack{M\to\infty \\ M \text{ even/odd}}} \mathbb{E}_{\mathrm{SO(even/odd)}}[(Z_\phi(U)-\mu_\pm)^n]$$

where

$$\mu_\pm := \lim_{\substack{M\to\infty \\ M \text{ even/odd}}} \mathbb{E}_{\mathrm{SO(even/odd)}}[Z_\phi(U)].$$

Katz-Sarnak Density Conjecture

Conjecture (Katz-Sarnak Density Conjecture)

The scaling limits of the distributions of zeros of families of automorphic L-functions agree with the scaling limits of eigenvalue distributions of classical subgroups of the unitary groups $U(N)$.

For our family of orthogonal L-functions, this posits

$$\lim_{N\to\infty} \langle (D(f;\phi) - \langle D(f;\phi)\rangle_{\pm})^n)\rangle_{\pm}$$
$$= \lim_{\substack{M\to\infty \\ M \text{ even/odd}}} \mathbb{E}_{(\mathrm{SO(even/odd)})}[(Z_\phi(U) - \mu_{\pm})^n]$$

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$$\lim_{N\to\infty} \langle (D(f;\phi) - \langle D(f;\phi)\rangle_{\pm})^n)\rangle_{\pm} = \lim_{\substack{M\to\infty \\ M \text{ even/odd}}} \mathbb{E}_{(\mathrm{SO(even/odd)})}[(Z_\phi(U) - \mu_{\pm})^n]$$

Within $\mathrm{supp}(\hat{\phi}) \in (-\frac{1}{n}, \frac{1}{n})$, $\langle (D(f;\phi) - \langle D(f;\phi)\rangle_{\pm})^n)\rangle_{\pm}$ look the same.

Iwaniec, Luo, and Sarnak's work

Theorem (Iwaniec, Luo, Sarnak, 2000)

Let ϕ be an even Schwartz test function with support of $\hat{\phi}$ in $(-2,2)$. Then for a family of cuspidal newforms of weight k and level N,

$$\lim_{N\to\infty} \langle D(f;\phi)\rangle_+ = \lim_{\substack{M\to\infty \\ M \text{ even}}} \mathbb{E}_{\mathrm{SO(even)}}[Z_\phi(U)]$$

$$\lim_{N\to\infty} \langle D(f;\phi)\rangle_- = \lim_{\substack{M\to\infty \\ M \text{ odd}}} \mathbb{E}_{\mathrm{SO(odd)}}[Z_\phi(U)]$$

where SO(even) is the scaling limit of special orthogonal even matrices and SO(odd) is the scaling limit of special orthogonal odd matrices.

Hughes' and Miller's work

Theorem (Hughes, Miller, 2006)

Let $n \geq 2, \text{supp}(\hat{\phi}) \subset \left(-\frac{1}{n-1}, \frac{1}{n-1}\right)$. Then

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} \langle (D(f;\phi) - \langle D(f;\phi)\rangle_{\pm})^n)\rangle_{\pm}$$

$$= \lim_{\substack{M\to\infty \\ M \text{ even/odd}}} \mathbb{E}_{(\mathrm{SO(even/odd)})}[(Z_\phi(U) - \mu_{\pm})^n]$$

Consequences for order of vanishing

- By optimally picking ϕ, agreement of the Katz-Sarnak density conjecture allows us to lower bound what fraction of $L(s, f)$ have a fixed order of vanishing at $s = 1/2$.
- Larger support of $\hat{\phi}$ allows us to localize ϕ more and get better estimates near the central point.

Order of vanishing bounds for SO(even)

Order vanishing	1-level	2-level	4^{th} centered moment*
6	0.144090	0.0157687	0.008538410
8	0.108067	0.0157687	0.000813368
10	0.086454	0.0047306	0.000186846

- These are upper bounds for vanishing at least r (number in order vanishing column).
- For the 1-level column, used the optimal 1-level bound from [ILS]. The support of the Fourier transform of the test function used is $(-2,2)$.
- For the 2-level column, used the optimal 2-level bound from [BCDMZ]. The support of the Fourier transform of the test functions is $(-1,1)$.
- For the 4^{th} centered moment* column, the * signifies that we used the 4 copies of the naive test functions φ_{naive}. The support of the Fourier transform of the test function used is $(-1/3,1/3)$.

Order of vanishing bounds for SO(odd)

Order vanishing	1-level	2-level	4^{th} centered moment*
5	0.222908	0.0674429	0.06580440
7	0.159220	0.0299746	0.00221997
9	0.123838	0.0168607	0.00036405

- These are upper bounds for vanishing at least r (number in order vanishing column).
- For the 1-level column, using the optimal 1-level bound from [ILS]. The support of the Fourier transform of the test function used is $(-2, 2)$.
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- For the 4^{th} centered moment* column, the * signifies that we used the 4 copies of the naive test functions φ_{naive}. The support of the Fourier transform of the test function used is $(-1/3, 1/3)$.

Our Work

Goal

For $L(s,f)$ and $a \leq n/2$, if $\text{supp}(\hat{\phi}) \subset (-\frac{1}{n-a}, \frac{1}{n-a})$, then the n^{th} centered moments of $D(f;\phi)$ averaged over $f \in H_k^{\pm}$ satisfy the Density Conjecture.

We generalized many number-theoretic techniques to explicitly calculate the n^{th} centered moments of $D(f;\phi)$ for this support.

Our assumptions

Hypothesis (Generalized Riemann Hypothesis, $L(s,\chi)$)

For every Dirichlet character χ, every nontrivial root s of $L(s,\chi)$ must have $\mathrm{Re}(s) = 1/2$.

Hypothesis (Generalized Riemann Hypothesis, $L(s,f)$)

For every cuspidal newform of weight k, level N, every nontrivial root s of $L(s,f)$ must have $\mathrm{Re}(s) = 1/2$.

Methods: the term of interest

To compute the n^{th} centered moments of $D(f;\phi)$, it suffices to convert the following sum over primes into a sum of integrals:

$$S_2^{(n)} := \sum_{p_1 \nmid N, \dots, p_n \nmid N} \prod_{j=1}^{n} \left(\hat{\phi}\left(\frac{\log p_j}{\log R} \right) \frac{2 \log p_j}{\sqrt{p_j} \log R} \right) \langle \lambda_f(N) \prod_{j=1}^{n} \lambda_f(p_i) \rangle_*,$$

where $\lambda_f(p_i)$ are the normalized Fourier coefficients of f.

- We explicitly calculate the term by applying the Petersson formula to convert Fourier coefficients into Bessel-Kloosterman terms.
- We remove negligible subterms before converting the rest into integrals.

Methods: reducing the term

Expand Fourier coefficients into exponential (*Bessel-Kloosterman*) sums

↓

Expand exponential sums into terms involving Dirichlet characters and Gauss sums

↓

Remove non-principal characters to get Ramanujan sums

↓

Convert sums over integers into integrals of $\widehat{\phi^k}$

Combinatorial obstructions

Subterms of $S_2^{(n)}$, where $\sum m_j \leq n, m_j \equiv n_j \pmod 2$:

$$i^k \sqrt{N} \sum_{\substack{q_1 \nmid N, \cdots, q_\ell \nmid N \\ q_j \text{ distinct}}} \prod_{j=1}^{\ell} \left(\hat{\phi} \left(\frac{\log p_j}{\log R} \right)^{n_j} \left(\frac{2 \log p_j}{\sqrt{p_j} \log R} \right)^{n_j} \right) \langle \lambda_f (N q_1^{m_1} \cdots q_\ell^{m_\ell}) \rangle$$

- As a increases more terms emerge and closed form for combinatorics in general case is difficult.
- We eliminate subterms of $S_2^{(n)}$ via combinatorics and the Weil bound.

Example: Weil bound application

Lemma

Suppose $\text{supp}(\hat{\phi}) \subseteq \left(-\frac{1}{n-a}, \frac{1}{n-a}\right)$, and assume GRH for $L(s,f)$. If $\ell \leq n-a$, the above subterm is $O(N^{-\varepsilon})$.

Lemma

Suppose $\text{supp}(\hat{\phi}) \subseteq \left(-\frac{1}{n-a}, \frac{1}{n-a}\right)$, and assume GRH for $L(s,f)$. If for at least $\ell + a - n$ primes we have $n_i > m_i$, the above subterm is $O(N^{-\varepsilon})$.

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More Combinatorial Obstructions

$$\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{q_1\nmid N,\cdots,q_\ell\nmid N} \sum_{m\le N^\varepsilon} \frac{1}{m} \sum_{b=1}^{\infty} \frac{S(m^2, Nq_1^{m_1}\cdots q_\ell^{m_\ell}; Nb)}{b\sqrt{N}}$$

$$\times J_{k-1}\left(\frac{4\pi m\sqrt{q_1^{m_1}\cdots q_\ell^{m_\ell}}}{b\sqrt{N}}\right) \prod_{j=1}^{\ell}\left(\hat{\phi}\left(\frac{\log q_j}{\log R}\right)^{n_j}\left(\frac{2\log q_j}{\sqrt{q_j}\log R}\right)^{n_j}\right)$$

- Convert Kloosterman sum to Gauss sums over $\chi \pmod b$

$$S(m^2, NQ; Nb) = -\frac{1}{\varphi(b)} \sum_{\chi \,(\mathrm{mod}\ b)} G_\chi(m^2) G_\chi((Q, b^\infty)) \overline{\chi}\left(\frac{Q}{(Q, b^\infty)}\right)\chi(N)$$

More Combinatorial Obstructions

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$$\times J_{k-1}\left(\frac{4\pi m \sqrt{q_1^{m_1} \ldots q_\ell^{m_\ell}}}{b\sqrt{N}}\right) \prod_{j=1}^{\ell} \left(\hat{\phi}\left(\frac{\log q_j}{\log R}\right)^{n_j} \left(\frac{2\log q_j}{\sqrt{q_j}\log R}\right)^{n_j}\right)$$

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- Case on which primes among $q_1, \ldots, q_\ell$ divide b

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Convert sums over integers into integrals of $\widehat{\phi^k}$

The limit $2/n$

- We show that non-principal characters in our sum do not contribute in the limit for support $(-2/n, 2/n)$. It turns out to be natural support for the random matrix theory equivalent.
- We need an application of GRH for Dirichlet L-functions:

$$\sum_{p \le x} \chi(p) p^{it} \log p = O\left(x^{1/2} (bxt)^{\varepsilon}\right)$$

for non-principal character χ modulo b and any $\varepsilon > 0$.

Proof Outline

Via the inverse Mellin transform of the Bessel function and shifting the contour,

$$\sum_{p_1,\dots,p_n<R^\sigma} J_{k-1}\left(\frac{4\pi m\sqrt{p_1\dots p_n}}{b\sqrt{N}}\right)\prod_{j=1}^{n}\left(\hat{\phi}\left(\frac{\log p_j}{\log R}\right)\frac{\log p_j\overline{\chi}(p_j)}{\sqrt{p_j}\log R}\right)$$

$$= \frac{1}{2\pi i}\int_{\Re(s)=1}\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)\left(\sum_{p<R^\sigma}\hat{\phi}\left(\frac{\log p}{\log R}\right)\frac{\log p\overline{\chi}(p)}{p^{(1+s)/2}\log R}\right)^n ds$$

$$\ll \frac{m}{b\sqrt{N}}R^{\sigma n/2+\varepsilon}. \qquad \text{(apply previous character sum)}$$

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$$\ll \frac{m}{b\sqrt{N}}R^{\sigma n/2+\varepsilon}. \qquad \text{(apply previous character sum)}$$

So the entire Bessel-Kloosterman term is

$$\ll N^{-1/2}\sum_{m\le N^\varepsilon}\frac{1}{m}\sum_{\substack{(b,N)=1\\ b<N^{2006}}}\frac{1}{b\varphi(b)}\sum_{\substack{\chi(b)\\ \chi\neq\chi_0}}|G_\chi(m^2)G_\chi(1)|\left(\frac{m}{b\sqrt{N}}R^{\sigma n/2+\varepsilon}\right)$$

$$\ll N^{-1+\sigma n/2+\varepsilon}$$

Methods: reducing the term

Expand Fourier coefficients into exponential (*Bessel-Kloosterman*) sums

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Convert sums over integers into integrals of $\widehat{\phi^k}$

Sums to Integrals

Under the Riemann Hypothesis for $\zeta(s)$, if $supp(\hat{\phi}) \subset \left(-\frac{1}{n-a}, \frac{1}{n-a}\right)$, then, for $0 \le \beta \le a/2$, and for any N,

$$\sum_{v_1,\dots,v_{n-\beta}} \left[\prod_{i=1}^{n-\beta} \hat{\phi}\left(\frac{\log v_i}{\log R}\right)\left(\frac{\chi_0(v_i)\Lambda(v_i)}{\sqrt{v_i}\log R}\right)\right] J_{k-1}\left(\frac{4\pi m\sqrt{v_1\dots v_{n-\beta}}}{b\sqrt{N}}\right)$$

$$= \sum_{c=0}^{a-\beta-1}\sum_{j=c}^{a-\beta-1}(-1)^{j-c}\binom{n-\beta}{j}\binom{j}{c}\frac{b\sqrt{N}}{2\pi m\log R}$$

$$\times \sum_{r_1,\dots,r_c=1}^{\infty}\left[\prod_{i=1}^{c}\hat{\phi}\left(\frac{\log r_i}{\log R}\right)\frac{\chi_0(r_i)\Lambda(r_i)}{r_i\log R}\right]$$

$$\times \int_{x=0}^{\infty} J_{k-1}(x)\widehat{\Phi_{n-\beta-c}}\left(\frac{2\log(bx\sqrt{N/\prod_{i=1}^{c} r_i}/4\pi m)}{\log R}\right)dx + O(N^{-\varepsilon}).$$

Evaluating the Main Term

$$\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{p_1,\ldots,p_n N} \prod_{j=1}^{n} \hat{\phi}\left(\frac{\log p_j}{\log R}\right) \frac{\log p_j}{\sqrt{p_j}\log R} \langle \lambda_f(Np_1\ldots p_n)\rangle_*$$

$$= -\sum_{\nu=0}^{a-1} \binom{n}{\nu} \sum_{i=0}^{a-\nu-1} (-1)^i \binom{n-\nu}{i} \frac{2^n}{\log R} \sum_{p_1,\ldots,p_\nu} \sum_{m\le N^\varepsilon} \frac{1}{m^2} \sum_{\substack{(b,N)=1 \\ b<N^2}} \frac{R(m^2,b)R(p_1\ldots p_\nu,b)}{\varphi(b)}$$

$$\times \int_{x=0}^{\infty} J_{k-1}(x) \widehat{\Phi_{n-\nu}} \left(\frac{2\log(bx\sqrt{N}/(4\pi m\sqrt{p_1\ldots p_\nu}))}{\log R}\right) dx \prod_{j=1}^{\nu} \hat{\phi}\left(\frac{\log p_j}{\log R}\right) \frac{\log p_j}{p_j \log R}.$$

- Largest term within $S_2^{(n)}$ and also the simplest to work with.
- Need to simplify Ramanujan sums to show agreement with random matrix theory.

Future Work

- Our work fully generalized the number theoretic techniques needed to convert these sums over primes into sums over integers.
- We are currently generalizing the work of Iwaniec, Luo, and Sarnak to convert these terms to integrals in order to show agreement with random matrix theory.

References

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[Bro17] Daniel Brox, *The riemann hypothesis and emergent phase space*, Journal of Modern Physics **08** (2017), 459–482.

[HM05] Christopher Hughes and Steven Miller, *Low lying zeros of l-functions with orthogonal symmetry*, Duke Mathematical Journal **136** (2005).

[ILS99] H. Iwaniec, W. Luo, and P. Sarnak, *Low lying zeros of families of l-functions*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **91** (1999), 55–131.

[Mon73] H. Montgomery, *The pair correlation of zeroes of the zeta function*, Proceedings of Symposia in Pure Mathematics **24** (1973), 181–193.

[RS96] Z. Rudnick and P. Sarnak, *Zeros of principal l-functions and random matrix theory*, Duke Mathematical Journal **81** (1996), 269–322.

Appendix A: Eliminating Ramanujan sums

Theorem (ILS)

Let ψ be an even Schwartz function with $\mathrm{supp}(\widehat{\psi}) \subset (-2,2)$. Then

$$\sum_{m \leq N^{\varepsilon}} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y) \widehat{\psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right) \frac{\mathrm{d}y}{\log R}$$
$$= -\frac{1}{2}\left[\int_{-\infty}^{\infty} \psi(x) \frac{\sin 2\pi x}{2\pi x} \mathrm{d}x - \psi(0)\right] + O\left(\frac{k \log\log kN}{\log kN}\right)$$

where $R = k^2 N$ and φ is Euler's totient function.

Lemma

Assume GRH for $L(s,f)$. If $\operatorname{supp}(\phi) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$ then

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} S_2^{(n)} = 2^{n-1}\left[\int_{-\infty}^{\infty} \phi(x)^n \frac{\sin 2\pi x}{2\pi x} dx - \frac{1}{2}\phi(0)^n\right]$$

$$-n2^{n-1}\left[\int_{-\infty}^{\infty} \hat{\phi}(x_2) \int_{-\infty}^{\infty} \phi(x_1)^{n-1} \frac{\sin(2\pi x_1(1+|x_2|))}{2\pi x_1} dx_1 dx_2 - \frac{1}{2}\phi(0)^n\right].$$