Determining Optimal Test Functions for the 2-Level Densities of *L*-functions

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Random Matrix Theory (RMT)

- Ensembles of matrices (e.g. real symmetric, Hermitian) with entries drawn from probability distribution.
- Study distribution of normalized eigenvalues for given ensemble.

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Katz-Sarnak density conjecture

Behavior of zeros of L-functions associated with cuspidal new forms well-modeled by RMT models.

1-level Density

Assume Riemann hypothesis for $L(s, f)$. Then there are are order $\log c_f$ low-lying non-trivial zeros $s = 1/2 + i\gamma_f$.

Test functions

We say $\phi : \mathbb{R}^n \to [0, \infty)$ is a test function if it is even, Schwartz class, Fourier transform $\widehat{\phi}$ compactly supported, and $\phi(0) > 0$.

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1-level density

$$D(f;\phi) := \sum_{\gamma_f} \phi\left(\frac{\gamma_f}{2\pi}\log(c_f)\right)$$

This measures the density of zeros of $L(s, f)$ near the central point $s = 1/2$.

1-level Density

Impossible to calculate $D(f;\phi)$ explicitly in practice, so take averages over finite subsets of $\mathcal{F}$:

$$\mathcal{F}(Q) := \{f \in \mathcal{F} : c_f \leq Q\}$$

$$\mathbb{E}(\mathcal{F}(Q);\phi) := \frac{1}{|\mathcal{F}(Q)|}\sum_{f\in\mathcal{F}} D(f;\phi)$$

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1-level density theorem

If $\mathcal{F}$ is complete in a certain spectral sense, then

$$\lim_{Q \to \infty} \mathbb{E}(\mathcal{F}(Q);\phi) = \int_{\mathbb{R}} \phi(x) W(\mathcal{F})(x)\, \mathrm{d}x$$

where $W(\mathcal{F})$ is a distribution depending on $\mathcal{F}$.

1-level Density

From RMT, $W(\mathcal{F})$ is dependent on a symmetry group $G = G(\mathcal{F})$ of $\mathcal{F}$. Write $W(\mathcal{F}) = W_{1,G}$.

1-level Density

From RMT, $W(\mathcal{F})$ is dependent on a symmetry group $G = G(\mathcal{F})$ of $\mathcal{F}$. Write $W(\mathcal{F}) = W_{1,G}$. Some examples:

$$W_{1,\mathrm{O}}(x) = 1 + \frac{1}{2}\delta(x)$$

$$W_{1,\mathrm{SO(Even)}}(x) = 1 + \frac{\sin(2\pi x)}{2\pi x}$$

$$W_{1,\mathrm{SO(Odd)}}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x} + \delta(x)$$

$$W_{1,\mathrm{Sp}}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x}$$

n-level Density

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$$D_n(f;\Phi) := \sum_{\substack{\gamma_{j,f} \\ |j| \text{ distinct}}} \Phi\left(\frac{\gamma_{1,f}}{2\pi}\log(c_f), \ldots, \frac{\gamma_{n,f}}{2\pi}\log(c_f)\right)$$

where $\Phi : \mathbb{R}^n \to [0,\infty)$ is a test function.

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$$\lim_{Q\to\infty} \mathbb{E}_n(\mathcal{F}(Q);\Phi) = \int_{\mathbb{R}^n} \Phi(x) W_n(\mathcal{F})(x_1,\dots,x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n$$

where $W_n(\mathcal{F})$ is a distribution depending on $\mathcal{F}$.

Katz-Sarnak Determinants

Set $K_\epsilon(x,y) := \frac{\sin(\pi(x-y))}{\pi(x-y)} + \epsilon\frac{\sin(\pi(x+y))}{\pi(x+y)}$, $\epsilon \in \{0, \pm 1\}$. n-level weights for classical compact groups are:

$$W_{n,\mathrm{SO(Even)}}(x) = \det\left(K_1(x_i,x_j)\right)_{i,j\le n}$$

$$W_{n,\mathrm{SO(Odd)}}(x) = \det\left(K_{-1}(x_i,x_j)\right)_{i,j\le n} + \sum_{k=1}^{n}\delta(x_k)\det\left(K_{-1}(x_i,x_j)\right)_{i,j\ne k}$$

$$W_{n,\mathrm{O}}(x) = \frac{1}{2}W_{n,\mathrm{SO(Even)}}(x) + \frac{1}{2}W_{n,\mathrm{SO(Odd)}}(x)$$

$$W_{n,\mathrm{U}}(x) = \det\left(K_0(x_i,x_j)\right)_{i,j\le n}$$

$$W_{n,\mathrm{Sp}}(x) = \det\left(K_{-1}(x_i,x_j)\right)_{i,j\le n}$$

The optimization problem

Quantities of interest

- $\text{AveRank}(\mathcal{F}(Q))$, the average order of vanishing at the central point $s = \frac{1}{2}$ for $\mathcal{F}(Q)$.
- $\text{WeightedAveRank}(\mathcal{F}(Q))$, the weighted average order of vanishing at the central point for $\mathcal{F}(Q)$.

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Bounds and an optimization problem

$$\lim_{Q\to\infty} \text{AveRank}(\mathcal{F}(Q)) \leq \frac{\int_{\mathbb{R}} \phi(x) W_{1,G}(x)\, \mathrm{d}x}{\phi(0)}$$

$$\lim_{Q\to\infty} \text{WeightedAveRank}(\mathcal{F}(Q)) \leq \frac{\int_{\mathbb{R}^n} \Phi(x) W_{n,G}(x)\, \mathrm{d}x}{\Phi(0)}$$

We want to minimize the right hand sides.

Old results: 1-level densities

Main Idea

Reduce the optimization problem to a differential equations problem via functional analysis.

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Main Result 1 [ILS00], [FM15]

A 1-level optimal test function ϕ for $\operatorname{supp} \widehat{\phi} \subseteq [-2\sigma, 2\sigma]$ exists for each classical compact group.

New results: 2-level densities

Main Idea

Restrict domain to only those test functions which are products of single variable test functions $\Phi(x, y) = \phi_1(x)\phi_2(y)$ for fixed admissible $\phi_1(x)$.

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Main Result 2

Can view the problem as a one-variable integration of ϕ_2 against a function of the form $\delta + m$, i.e. analogous to 1-level case.

Fixed test function and resulting estimates

Let

$$\phi_1(x) = \left(\frac{\sin(2\pi x)}{2\pi x}\right)^2.$$

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Family	Naive	Optimal
SO(E)	$\frac{5}{12} \approx 0.4166$	$\frac{1}{96}\left(54\sqrt{3}\cot\left(\frac{2}{\sqrt{3}}\right) - 5\right) \approx 0.3784$
SO(O)	$\frac{13}{12} \approx 1.0833$	$\frac{1}{32}\left(33 + 2\sqrt{3}\cot\left(\frac{2}{\sqrt{3}}\right)\right) \approx 1.079$
O	$\frac{3}{4} \approx 0.75$	$\frac{1}{24}\left(13 + 6\sqrt{3}\cot\left(\frac{2}{\sqrt{3}}\right)\right) \approx 0.733$
U	$\frac{1}{2} \approx 0.5$	$\frac{1}{12}(4 + 3\cot(1)) \approx 0.4939$
Sp	$\frac{1}{12} \approx 0.0833$	$\frac{1}{32}(3 + 2\cot(2)) \approx 0.06515$

Applications to Order of Vanishing

$\Pr(N) :=$ probability that $L(s,f)$ has zero of order N at $s = 1/2$.

$$\sum_{N=2}^{\infty} 2\Pr(N) \leq \sum_{N=0}^{\infty} N(N-1)\Pr(N) \leq \frac{\int_{\mathbb{R}^2} \Phi(x,y) W_{2,\mathcal{G}}(x,y)\, \mathrm{d}x\mathrm{d}y}{\Phi(0,0)}$$

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$$\Pr(0)+\Pr(1) \geq \begin{cases} 1 - \frac{13+6\sqrt{3}\cot\left(\frac{2}{\sqrt{3}}\right)}{48} \approx 0.633493 & W_{2,\mathrm{O}} \\ 1 - \frac{54\sqrt{3}\cot\left(\frac{2}{\sqrt{3}}\right)-5}{192} \approx 0.810776 & W_{2,\mathrm{SO(Even)}} \\ 1 - \frac{33+2\sqrt{3}\cot\left(\frac{2}{\sqrt{3}}\right)}{64} \approx 0.460457 & W_{2,\mathrm{SO(Odd)}} \\ 1 - \frac{4+3\cot(1)}{24} \approx 0.753072 & W_{2,\mathrm{U}} \\ 1 - \frac{3+2\cot(2)}{64} \approx 0.967427 & W_{2,\mathrm{Sp}} \end{cases}$$

Key observations

Observation 1

Ahiezer's Theorem and the Paley-Wiener theorem give a correspondence between test functions and L^2

$$\phi \text{ test function with } \operatorname{supp}(\hat{\phi}) \subseteq [-2\sigma, 2\sigma]$$

$$\Updownarrow$$

$$\widehat{\phi}(\xi) = (g * \breve{g})(\xi) \text{ for } g \in L^2[-\sigma, \sigma]$$

where $\breve{g}(\xi) = \overline{g(-\xi)}$.

Key observations

Observation 2

By Plancharel's theorem

$$\frac{\int_{\mathbb{R}} \phi(x) W_{1,G}(x)\, \mathrm{d}x}{\phi(0)} = \frac{\int_{\mathbb{R}} \hat{\phi}(\xi) \widehat{W_{1,G}}(\xi)\, \mathrm{d}\xi}{\int_{\mathbb{R}} \hat{\phi}(\xi)\, \mathrm{d}\xi}.$$

Observation 3

The Fourier transforms of the 1-level distributions $W_{1,G}$ take the form

$$\widehat{W_{1,G}}(\xi) = \delta(\xi) + m_G(\xi)$$

where m_G is a real-valued even step function.

Step 1: Convert to minimization over L^2

Define compact, self-adjoint linear operator
$K : L^2[-\sigma, \sigma] \to L^2[-\sigma, \sigma]$

$$(Kg)(x) := \int_{-\sigma}^{\sigma} m(x-y)g(y)\,\mathrm{d}y.$$

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Let $\operatorname{supp} \widehat{\phi} \subseteq [-2\sigma, 2\sigma]$, then

$$\frac{\int_{\mathbb{R}} \hat{\phi}(\xi)\widehat{W_{1,G}}(\xi)\,\mathrm{d}\xi}{\int_{\mathbb{R}} \hat{\phi}(\xi)\,\mathrm{d}\xi} = \frac{\int_{\mathbb{R}} (g * \breve{g})(\xi)(\delta(\xi) + m(\xi))\,\mathrm{d}\xi}{\int_{\mathbb{R}} (g * \breve{g})(\xi)\,\mathrm{d}\xi}$$

Step 1: Convert to minimization over L^2

$$
\begin{aligned}
&= \frac{\langle g, g\rangle_{L^2} + \int_{-\sigma}^{\sigma}\int_{-\sigma}^{\sigma} m(\xi - y) g(y)\,\mathrm{d}y\,\overline{g(\xi)}\,\mathrm{d}\xi}{|\langle \mathbf{1}, g\rangle_{L^2}|^2} \\
&= \frac{\langle g, g\rangle_{L^2} + \langle Kg, g\rangle_{L^2}}{|\langle \mathbf{1}, g\rangle_{L^2}|^2} \\
&= \frac{\langle (I+K)g, g\rangle_{L^2}}{|\langle \mathbf{1}, g\rangle_{L^2}|^2}.
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$$= \frac{\langle g, g\rangle_{L^2} + \int_{-\sigma}^{\sigma}\int_{-\sigma}^{\sigma} m(\xi - y) g(y)\,\mathrm{d}y\overline{g(\xi)}\,\mathrm{d}\xi}{|\langle \mathbf{1}, g\rangle_{L^2}|^2}$$

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$$= \frac{\langle (I+K)g, g\rangle_{L^2}}{|\langle \mathbf{1}, g\rangle_{L^2}|^2}.$$

Equivalent optimization problem

Minimize the functional $R : L^2[-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ given by

$$R(g) := \frac{\langle (I+K)g, g\rangle_{L^2}}{|\langle \mathbf{1}, g\rangle_{L^2}|^2}.$$

Step 2: Fredholm theory

Some observations:

- $R(g) \geq \lim_{Q\to\infty} \mathrm{AveRank}(\mathcal{F}(Q)) \geq 0$.
- Spectral Theorem $\Longrightarrow$ orthonormal basis of eigenvectors of K, eigenvalues λ_j.
- $\lambda_j \geq -1$.

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- $R(g) \geq \lim_{Q\to\infty} \text{AveRank}(\mathcal{F}(Q)) \geq 0$.
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Case 1: Eigenvalue (-1)

Let $f_0 \in L^2[-\frac{1}{2}, \frac{1}{2}]$ not orthogonal to 1 and $(I+K)f_0 = 0$,

$$R(f_0) = \frac{\langle (I+K)f_0, f_0\rangle_{L^2}}{|\langle \mathbf{1}, f_0\rangle_{L^2}|^2} = 0.$$

Step 2: Minimization

Case 2: $\lambda_j > -1$ for all j

More functional analysis!

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- $\ker(I + K) = \{0\}$ (all eigenvalues > -1).
- By Fredholm Theory, exists unique $f_0 \in L^2[-\frac{1}{2}, \frac{1}{2}]$ satisfying $(I + K)f_0 = \mathbf{1}$.
- $A := \langle \mathbf{1}, f_0 \rangle = \langle (I + K)f_0, f_0 \rangle_{L^2} > 0$.

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- $A := \langle \mathbf{1}, f_0 \rangle = \langle (I + K)f_0, f_0 \rangle_{L^2} > 0$.

For $g = f_0 + h \in L^2[-\frac{1}{2}, \frac{1}{2}]$ with $\langle \mathbf{1}, g \rangle_{L^2} \neq 0$, WLOG $\langle \mathbf{1}, g \rangle_{L^2} = A$. Then $\langle \mathbf{1}, h \rangle_{L^2} = 0$, so

$$R(g) = \frac{\langle \mathbf{1}, f_0 \rangle_{L^2} + \langle (I + K)h, h \rangle_{L^2} + \langle \mathbf{1}, h \rangle_{L^2} + \langle h, \mathbf{1} \rangle_{L^2}}{|A|^2}$$

$$= \frac{A + \langle (I + K)h, h \rangle_{L^2} + 0 + 0}{|A|^2} \geq \frac{1}{A} = R(f_0)$$

Challenges

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Larger support and higher level densities give better estimates on the average order of vanishing. Two main obstructions:

1. Small support does not detect non-constant kernels, e.g. $\widehat{W_{1,\mathrm{Sp}}}(x) = \delta(x) - \frac{1}{2}\mathbf{1}_{[-1,1]}(x)$.
2. $\widehat{W_{n,G}}$ more complicated and higher dimensional integral operators not as well-understood.

Simplifications

A restricted optimization problem

Fix a test function ϕ_1 with $\mathrm{supp}\,\widehat{\phi_1} \subseteq [-1,1]$, we want to minimize

$$\frac{\int_{[-1,1]^2} \hat{\phi}_1(\xi_1)\hat{\phi}_2(\xi_2)\widehat{W_{2,G}}(\xi)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2}{\phi_1(0)\phi_2(0)}$$

over test functions ϕ_2 with $\mathrm{supp}(\hat{\phi}_2) \subset [-1,1]$.

Example: Unitary

The 2-level distributions:

$$W_{2,\mathrm{U}}(x) = 1 - \frac{\sin^2(\pi(x_1 - x_2))}{\pi^2(x_1 - x_2)^2},$$

$$\widehat{W_{2,\mathrm{U}}}(\xi) = \delta(\xi_1)\delta(\xi_2) + \delta(\xi_1 + \xi_2)(|\xi_1| - 1)\mathbf{1}(\xi_1).$$

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For ϕ_1 arbitrary,

$$\begin{aligned}\tilde{V}_{\phi_1,\mathrm{U}}(\xi_2) &= \frac{1}{\phi_1(0)} \int_{\mathbb{R}} \hat{\phi}_1(\xi_2) \widehat{W_{2,\mathrm{U}}}(\xi)\, \mathrm{d}\xi_1 \\ &= \frac{\hat{\phi}_1(0)}{\phi_1(0)} \delta(\xi_2) + \frac{\hat{\phi}_1(-\xi_2)}{\phi_1(0)} (|\xi_2| - 1)\mathbf{1}(\xi_2) \\ &= c_{\phi_1,\mathrm{U}}\delta(\xi_2) + \tilde{m}_{\phi_1,\mathrm{U}}(\xi_2)\end{aligned}$$

Recovering the 1-level set-up

For each classical compact group G,

$$V_{\phi_1,G}(\xi_2) := \delta(\xi_2) + m_{\phi_1,G}(\xi_2), \qquad m_{\phi_1,G}(\xi_2) := \frac{\tilde{m}_{\phi_1,G}(\xi_2)}{c_{\phi_1,G}}.$$

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Optimization problem rehashed

Minimize

$$\frac{\int_{[-1,1]} \hat{\phi}_2(\xi_2) V_{\phi_1,G}(\xi_2) \mathrm{d}\xi_2}{\phi_2(0)}$$

over test functions ϕ_2 with $\operatorname{supp} \widehat{\phi_2} \subseteq [-1,1]$.

Collect the 1-level goodies

There exists a unique $g_{\phi_1,G} \in L^2[-1/2, 1/2]$ satisfying

$$1 = g_{\phi_1,G}(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} m_{\phi_1,G}(x-y) g_{\phi_1,G}(y)\,\mathrm{d}y.$$

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Moreover,

$$\frac{c_{\phi_1,G}}{\langle \mathbf{1}, g_{\phi_1,G} \rangle_{L^2}} = \inf_{\phi_2} \frac{\int_{[-1,1]} \hat{\phi}_2(\xi_2) V_{\phi_1,G}(\xi_2) \mathrm{d}\xi_2}{\phi_2(0)}.$$

Choosing ϕ_1

Natural choice of test function is the Fourier pair

$$\phi_1(x) = \left(\frac{\sin(2\pi x)}{2\pi x}\right)^2, \qquad \widehat{\phi_1}(\xi) = (1 - |\xi|)\mathbf{1}_{[-1,1]}(\xi)$$

Key observation

Kernels take the form of quadratic polynomials in $|x|$ on $[-1, 1]$, i.e.

$$m_{\phi_1,G}(x) = (a + b|x| + c|x|^2)\mathbf{1}_{[-1,1]}(x).$$

Differentiation under the integral

Exercise for the reader, order one term:

$$\frac{d}{dx}\int_{-1/2}^{1/2}|x-y|g(y)\mathrm{d}y = \int_{-1/2}^{x} g(y)\mathrm{d}y - \int_{x}^{1/2} g(y)\mathrm{d}y,$$

$$\frac{d^2}{dx^2}\int_{-1/2}^{1/2}|x-y|g(y)\mathrm{d}y = 2g(x)$$

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Order two term:

$$\frac{d}{dx}\int_{-1/2}^{1/2}(x-y)^2 g(y)\mathrm{d}y = \int_{-1/2}^{1/2}(2x-2y)g(y)\mathrm{d}y,$$

$$\frac{d^2}{dx^2}\int_{-1/2}^{1/2}(x-y)^2 g(y)\mathrm{d}y = 2\int_{-1/2}^{1/2} g(y)\mathrm{d}y.$$

Example: $W_{2,\mathrm{U}}$

We want to find $g \in L^2[-1/2, 1/2]$ obeying

$$1 = g(x) - \int_{-\frac{1}{2}}^{\frac{1}{2}} (1 - |x - y|)^2 g(y) \mathrm{d}y.$$

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$$0 = g''(x) - 4g(x) + 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} g(y) \mathrm{d}y$$

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Assuming evenness, solutions take the form

$$g(x) = A\cos(2x) + C.$$

Optimal $g_{\phi_1,G}$ for $\operatorname{supp}\widehat{\phi_1}, \operatorname{supp}\widehat{\phi_2} \subseteq [-1,1]$

$$g_{\phi_1,\mathrm{SO(Even)}}(x) = \frac{216\cos(4x/\sqrt{3}) + 36\sqrt{3}\sin(2/\sqrt{3})}{162\cos(2/\sqrt{3}) - 5\sqrt{3}\sin(2/\sqrt{3})},$$

$$g_{\phi_1,\mathrm{SO(Odd)}}(x) = \frac{8\cos(4x/\sqrt{3}) + 12\sqrt{3}\sin(2/\sqrt{3})}{11\sqrt{3}\sin(2/\sqrt{3}) + 2\cos(2/\sqrt{3})},$$

$$g_{\phi_1,\mathrm{U}}(x) = \frac{6\cos(2x) + 6\sin(1)}{3\cos(1) + 4\sin(1)},$$

$$g_{\phi_1,\mathrm{O}}(x) = \frac{36\cos(4x/\sqrt{3}) + 18\sqrt{3}\sin(2/\sqrt{3})}{18\cos(2/\sqrt{3}) + 13\sqrt{3}\sin(2/\sqrt{3})},$$

$$g_{\phi_1,\mathrm{Sp}}(x) = \frac{8\cos(4x) + 12\sin(2)}{2\cos(2) + 3\sin(2)}$$

Iterating ϕ_k

Some terminology

Fix a test function ϕ with $\operatorname{supp} \widehat{\phi_1} \subseteq [-1, 1]$. We say ψ is *optimal* for ϕ if ψ minimizes

$$\frac{\int_{[-1,1]^2} \phi(x)\psi(y) W_{2,G}(x, y) \, \mathrm{d}x \, \mathrm{d}y}{\phi(0)\psi(0)}$$

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- Recall correspondence between L^2 and test functions: optimal ϕ takes the form $g * \check{g}$.
- To improve bounds, we iterate ϕ_k: finding optimal ϕ_3 for ϕ_2, finding optimal ϕ_4 for ϕ_3, etc.

Example: Unitary

Define

$$(Kg)(x) := \int_{-\sigma}^{\sigma} m_{\mathsf{U},\phi_2}(x-y)g(y)\,\mathrm{d}y.$$

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Summing five terms gives the new bound

$$c_{G,\phi}\left(\sum_{n=0}^{5}(-1)^n \int_{-1/2}^{1/2} K^n(1)(x)dx\right)^{-1} \approx 0.4888$$

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