Erdős Distinct Angle Problems

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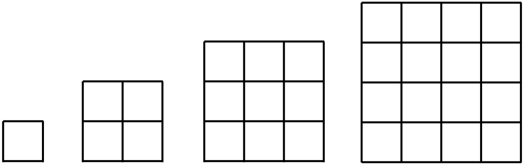
Erdős Distinct Distance Problems

- Erdős 1946:
  - *What are the asymptotic bounds on the minimum number of distinct distances among point sets with n points?*
- The integer lattice provides upper bound $O(n/\sqrt{\log n})$.
  - The number of positive integers smaller than n that are the sum of two squares is $\Theta(n/\sqrt{\log n})$ (Landau-Ramanujan).
- Only finally resolved in 2015 by Guth and Katz with a lower bound of $\Omega(n/\log(n))$.



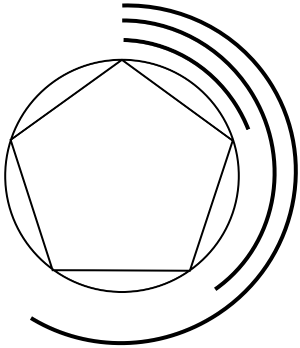
Variants of the Distance Problem

1. What is the minimal number of distinct distances among sets of n points with no three on a line or both no three on a line and no four on a circle?
2. What do "optimal" low distance configurations look like?
3. What is the maximal size of a k-distance set?
4. For a fixed n points, what is the largest subset with all distinct distances? In restricted settings?
5. What about point sets in higher dimensions?

There are many, many more. See Adam Sheffer's survey.

The Erdős Distinct Angle Problem

- Erdős and Purdy, 1995:
  - *What is the minimum number of distinct angles, $A(n)$, in $(0, \pi)$ formed by n non-collinear points in the plane?*
- They conjectured that regular n-gons are optimal ($n - 2$ distinct angles):



General Lower Bound on the Erdős Angle Problem

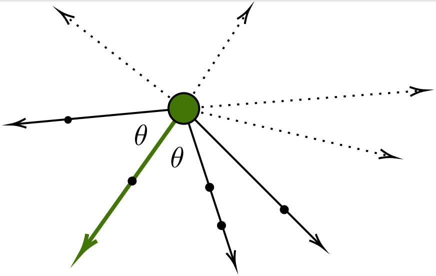
Conjecture (Weak Dirac Conjecture)

Every set $\mathcal{P}$ of n non-collinear points in the plane contains a point incident to at least $\lceil n/2 \rceil$ lines between points in $\mathcal{P}$.

The best current bound of $\left\lceil \frac{n}{3} \right\rceil + 1$ was proven by Han in 2017.

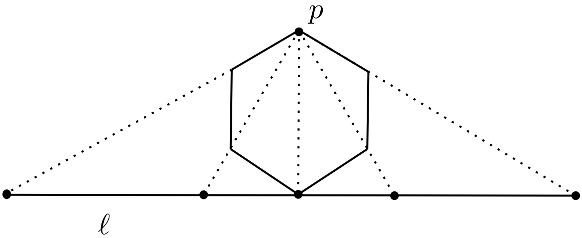
Corollary

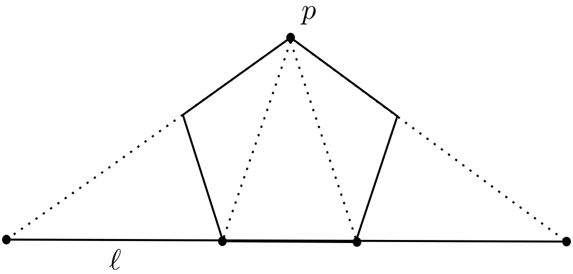
$A(n) \geq \frac{n}{6}$, $A_{no3l}(n) \geq \frac{n-2}{2}$.



Projected Polygon

What is the optimal configuration for $A_{no4c}(n)$, the minimum number of distinct angles among n points with no 4 cocircular?





General Position Bounds

Theorem

$A_{gen}(n) = O(n^{\log_2(7)})$.

Proof sketch:

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- Note that $p_1 - p_2 = p_3 - p_4 \implies |T(p_1) - T(p_2)| = |T(p_3) - T(p_4)|$.

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- **This means triangles congruent up to edge translation are congruent under the projection.**
- The number of equivalence classes of edge translation equivalent triangles in a d-dimensional cube is

$$\frac{7^d - 3^{d+1} + 2}{12}.$$

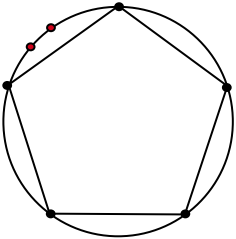
How much can an optimal configuration be perturbed while remaining "near-optimal?"

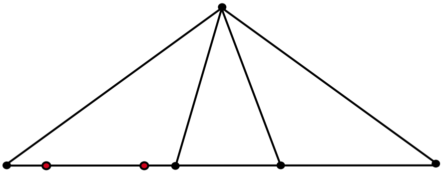
Theorem

The maximum number of distinct angles in configurations of a regular $(n-k)$-gon with k points placed on the same circle is $\Theta(nk)$.

Theorem

The maximum number of distinct angles in configurations of a projected $(n-k)$-gon with k points placed on the same projected line is $\Theta(nk)$.





Low Angle Constructions

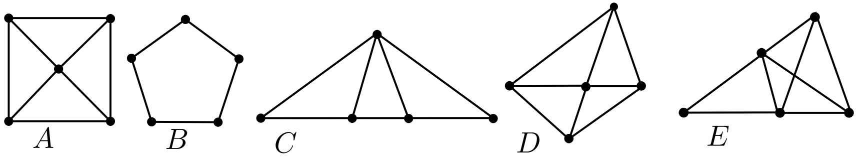
Definition

Let $P(k)$ be the maximum number of points in a planar point configuration admitting at most k distinct angles.

Lemma

$k + 2 \leq P(k) \leq 6k$.

- $P(2) = 5$. The unique optimal configuration is A.
- $P(3) = 5$. There are 5 unique optimal configurations.



What is the minimum maximum size of a subset of n points with no repeated angles, $R(n)$?

Lemma

Let $\mathcal{P} \subseteq \mathbb{R}^2$ such that $|\mathcal{P}| = n$ and $\mathcal{P}$ contains no 3 collinear points. $R(n) \leq (2A(\mathcal{P}))^{\frac{1}{3}}$.

- $S \subseteq \mathcal{P}$ admits at most $A(\mathcal{P})$ distinct angles.
- Moreover, if $3\binom{|S|}{3} > A(\mathcal{P})$, there are repeated angles in S.
- $\implies R(n), R_{\text{no3l}}(n) = O(n^{1/3})$
- Moreover, $R_{\text{no4c}}(n), R_{\text{gen}}(n) = O(n^{\log_2(7)/3})$.

Lower bound in general position #1

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$R_{gen}(n) = \Omega(n^{1/5})$.

- Let $\mathcal{P}$ be a point configuration in general position with n points.

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- Remove an element from $\mathcal{Q}$ in each of the pairs in the q_i-sets to form $\mathcal{Q}'$.
- $\mathbb{E}[|\mathcal{Q}'|] \geq pn - \sum_{i=3}^{6} p^i q_i(n)$.

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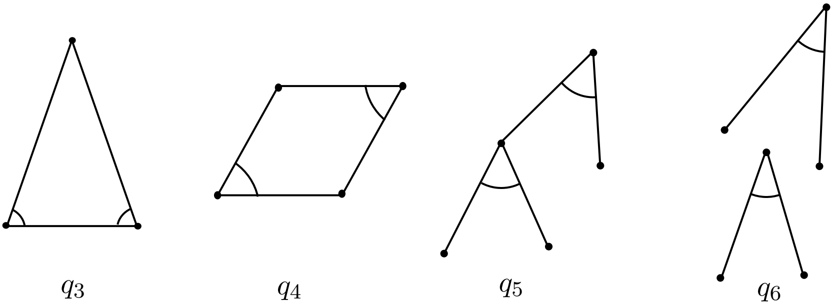
- $q_3(n) = O(n^{7/3}), q_4(n) = O(n^3), q_5(n) = O(n^4), q_6(n) = O(n^5)$.



Example configurations of $q_3(n), q_4(n), q_5(n), q_6(n)$.

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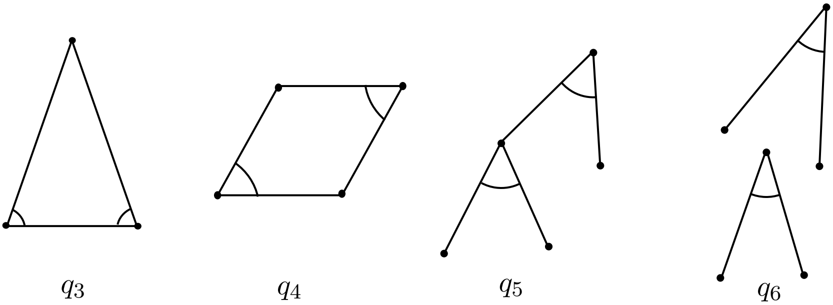

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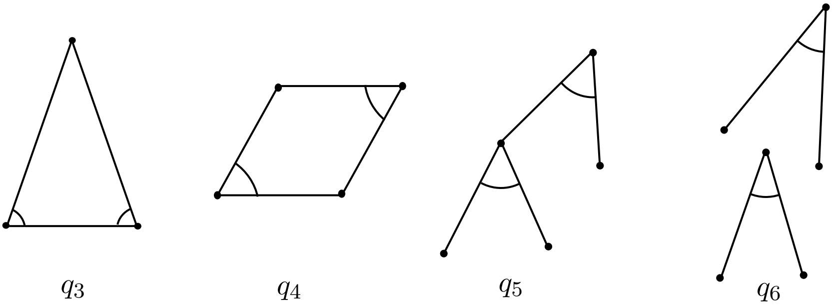

Example configurations of $q_3(n), q_4(n), q_5(n), q_6(n)$.

- Let $p = cn^{-4/5}$ for some carefully chosen constant c, and conclude the result!

Higher Dimensions: Lenz's Construction

Theorem

For $d \geq 6$, the smallest number of angles that may be defined by n points is at most $2\left\lceil \frac{n}{\lfloor d/2 \rfloor} \right\rceil - 2$, achieved by Lenz's Construction.

- Lenz's Construction consists of unit circles centered at the origin, arranged in disjoint pairs of dimensions.
- We divide the points evenly among the circles, and on each they form a regular polygon.
- $(\cos(i\pi/n), \sin(i\pi/n), 0, 0, \dots), (0, 0, \cos(j\pi/n), \sin(j\pi/n), \dots), \dots$

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