Closed form densities for the limiting spectral measure of random circulant Hankel matrices.

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August X, 2021

Origins of Random Matrix Theory

- Nuclear physics
    - In the 1950s Wigner was studying the energy spectrum of Uranium, and postulated that the eigenvalues of properly chosen families of matrices would correspond with the various quantized energy levels of the nucleus.
    - In order to understand the behavior of atomic nuclei Wigner had to understand the time-independent Schrodinger equation, which is given by

$$H\Psi_n = E_n\Psi_n$$

    - Of interest are the eigenvalues, but these are infinite dimensional matrices... How does one analyze such a system?

Wigner and Nuclear Physics

- Wigner's great insight was, rather than study the infinite dimensional system explicitly, instead construct families of $N \times N$ matrices with entries chosen from a given probability distribution approximating the system behavior, and then take the limit as N approaches infinity.
- Here the complexity of the system is used to our advantage, as the actual system is well approximated by the system average.

Motivation

- Many different ensembles of matrices have been investigated. Certain families have been found to densities of normalized eigenvalues given by nice distributions.
- Real symmetric Toeplitz matrices are *almost* Gaussian, but there are Diophantine obstructions. These issues are side-stepped by instead studying real symmetric palindromic Toeplitz matrices: the density is given by the Gaussian
- The distribution of real symmetric palindromic Hankel matrices is also given by the Gaussian
- However, as we will see the additional symmetry imposed by studying circulant matrices will lend itself to a different distribution

Concentric Even

$$\begin{pmatrix}
x_2 & x_1 & x_0 & x_3 & x_3 & x_0 & x_1 & x_2 \\
x_1 & x_0 & x_3 & x_2 & x_2 & x_3 & x_0 & x_1 \\
x_0 & x_3 & x_2 & x_1 & x_1 & x_2 & x_3 & x_0 \\
x_3 & x_2 & x_1 & x_0 & x_0 & x_1 & x_2 & x_3 \\
x_3 & x_2 & x_1 & x_0 & x_0 & x_1 & x_2 & x_3 \\
x_0 & x_3 & x_2 & x_1 & x_1 & x_2 & x_3 & x_0 \\
x_1 & x_0 & x_3 & x_2 & x_2 & x_3 & x_0 & x_1 \\
x_2 & x_1 & x_0 & x_3 & x_3 & x_0 & x_1 & x_2
\end{pmatrix}$$

Concentric Even Matrices

- Insert figure, describe delineations of the different quadrants of the concentric even matrix that correspond to swirl pattern
- They are called concentric even because there is the appearance of circles of equivalent entries, when looking more closely through the swirl lens, are permutations of a sequence of entries from 1 to n, or just layers of sequences of these entries from 1 to n
- comment on symmetry that has been built in to this ensemble and how this will influence the spectral distribution
- and then the even portion refers to the matrix being $2N \times 2N$, further contributing to the symmetry and allowing the reduction to studying circulant Hankel describe/show example/decompose

Matrix Swirl

- Generalized concentric even, the basis of the swirl idea is having four quadrants in a block matrix, where each quadrant is defined by the $N \times N$ inputs, A, X.
- -insert image
- show definition
- comment about specifically using exchange or permutation matrices
- Swirl theorems (in particular those that utilize $X^2 = I$ and the trace eigenvalue lemma. Of course the one that leads to the typical circulant Hankel matrix being the focus of study
- Inspired by Wigner we might week to study the swirl of a matrix with itself. When our matrix X is a permutation matrix, this problem reduces to studying the trace of AA^T.

Circulant Swirl

- Comment on the specific example of the circulant swirl ensemble, with circulant Toeplitz in the first quadrant and circulant Hankel in the second quadrant with the transitional exchange matrix

Spectral Measure

- The empirical spectral distribution of an $N \times N$ matrix A with eigenvalues $\lambda_1(A), \ldots, \lambda_N(A)$ is

$$\mu_{A,N}(x) := \frac{1}{c} \sum_{i=1}^{N} \delta \left(x - \frac{\sqrt{c}\lambda_i(A)}{N} \right). \tag{1}$$

- c is normalization factor and δ is Dirac delta function.
- This can be seen as putting a point mass at each normalized eigenvalue.

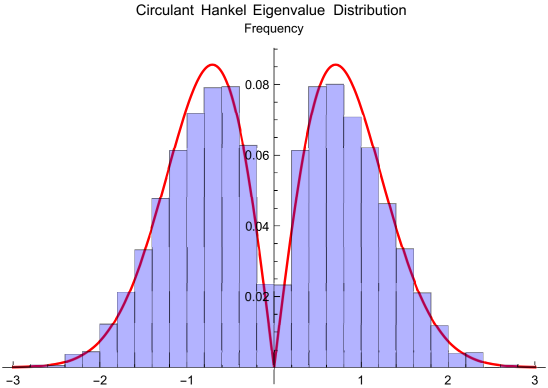


Figure: Histogram of eigenvalues for 100 40×40 random circulant Hankel matrices. A symmetric Rayleigh distribution is shown in red.

Moments of Spectral Measure

$$\mu_{A,N}(x) := \frac{1}{c}\sum_{i=1}^{N}\delta\left(x - \frac{\sqrt{c}\lambda_i(A)}{N}\right) \tag{2}$$

- We can express the kth moment of the empirical spectral distribution as:

$$M_k(A,N) := \int_{-\infty}^{\infty} x^k \mu_{A,N}(x)dx = \frac{c^{k/2-1}}{N^k}\sum_{i=1}^{N}\lambda_i^k(A). \tag{3}$$

- Want to calculate average/expectation of $M_k(A,N)$ for all A belonging to some random matrix ensemble.

Eigenvalue Trace Lemma

Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_1(A), \ldots \lambda_N(A)$. Then

$$\sum_{i=1}^{N} \lambda_i^k(A) = \mathrm{tr}(A^k).$$

Moments of Spectral Measure

$$M_k(A,N) = \frac{c^{k/2-1}}{N^k} \sum_{i=1}^{N} \lambda_i^k(A) \tag{4}$$

- Applying eigenvalue trace lemma to moments gives

$$M_k(A,N) = \frac{c^{k/2-1}}{N^k} \operatorname{tr}(A^k) \tag{5}$$

- Allows us to reduce problem to combinatorics.

Moments of Spectral Measure

$$M_k(A,N) = \frac{c^{k/2-1}}{N^k}\,\mathrm{tr}(A^k) \tag{6}$$

- For our ensemble, we take $c = N$ and calculate the expected value of the family:

$$M_k(N) := \mathbb{E}_{A\in H_N}\left[M_k(A,N)\right] = \frac{1}{N^{k/2+1}}\mathbb{E}_{A\in H_N}\left[\mathrm{tr}(A)^k\right]. \tag{7}$$

- Expectation is over family of $N \times N$ circulant Hankel matrices H_N.
- Taking $N \to \infty$ gives kth moment of limiting spectral distribution:

$$M_k = \lim_{N\to\infty} M_k(N). \tag{8}$$

1st and 2nd moments

Lemma

$M_1 = 0,\ M_2 = 1$

- $M_1 = \lim_{N\to\infty} \frac{N\mathbb{E}[x_i]}{N^{3/2}} = \lim_{N\to\infty} \frac{0}{N^{3/2}} = 0.$
- $M_2 = \lim_{N\to\infty} \frac{N^2\mathbb{E}[x_i^2]}{N^2} = 1.$

$$\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ x_3 & x_0 & x_1 & x_2 \\ x_2 & x_3 & x_0 & x_1 \\ x_1 & x_2 & x_3 & x_0 \end{bmatrix}$$

4×4 Circulant Hankel Matrix

Odd Moments

Theorem

$M_{2k+1} = 0$ *for all* $k \in \mathbb{Z}_{\geq 0}$.

Note:

$$\mathrm{tr}((H_N)^{2k+1}) = \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_{2k+1}=1}^{N} c_{i_1 i_2} c_{i_2 i_3} \ldots c_{i_{2k+1} i_1}.$$

- The summands of $\mathbb{E}[\mathrm{tr}(H_N)^k]$ that contribute must have x_i's matched in at least pairs.
- Thus, there are at most k matched groups.
- We can choose the x_i's for the groups in at most N^k ways.
- Specifying one index in N ways then specifies the rest up to less than k^{2k} (a constant in N) choices.
- So,

$$M_{2k+1} = \lim_{N \to \infty} \frac{O(N^{k+1})}{N^{k+3/2}} = 0.$$

Even Moments

Theorem

$M_{2k} = k!.$

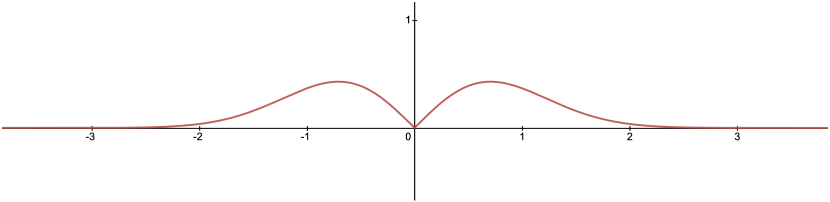
$$\mathrm{tr}((H_N)^{2k}) = \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_{2k}=1}^{N} c_{i_1 i_2} c_{i_2 i_3} \cdots c_{i_{2k} i_1}.$$

- Contributing summands must be matched in pairs.
- Choose the x_i involved in the summand in N^k ways and some initial index in another N.
- There are $k!$ such matchings (must be even-odd pairings only).
- So,

$$M_{2k} = \lim_{N\to\infty} \frac{k! N^{k+1} \prod_{i=1}^{k} \mathbb{E}[x_i^2]}{N^{k+1}} = k!.$$

Rayleigh Distribution

- $f(x) = |x|e^{-x^2}$ is the symmetrized Rayleigh distribution.
- The Rayleigh distribution naturally describes 2-D wind velocity.
- The $2k$th moment of the (symmetrized) Rayleigh distribution is $k!$ and the $2k+1$th is 0.
- Then, $\mu_{H_N,N}(x) \to f(x)$ $a.s$ as $N \to \infty$ ([1]).
- The same happens for swirl with circulant Toeplitz and exchange matrices.



Acknowledgements

This research was done as part of the SMALL REU program and was funded by NSF grant number 1947438.

Special thanks to our co-researchers Faye Jackson, Simran Khunger, Luke Reifenberg, and Professor Steven J. Miller for his mentorship.

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