

Extending support in calculating the 1-level density of low lying zeros of families of L -functions

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A surprising connection

The zeroes of L -functions near the central point and the energy levels of heavy nuclei can both be modeled by eigenvalues of random matrices.

The limiting distribution of eigenvalues of random matrices is reasonable to calculate, while the distribution of zeroes of L -functions is elusive.

Definition

Let $H_k^*(N)$ be the family of holomorphic cuspidal newforms of weight k and level N .

Definition (L -function associated to $f \in H_k^*(N)$)

$$L(s, f) := \sum_{n=1}^{\infty} \lambda_f(n) n^{-s},$$

where $\lambda_f(n) := a_f(n) n^{-(k-1)/2}$, for $a_f(n)$ the n^{th} Fourier coefficient.

The completed L -function $\Lambda(s, f)$ of $L(s, f)$ satisfies the following functional equation:

$$\Lambda(s, f) = \left(\frac{\sqrt{N}}{2\pi} \right)^s \Gamma\left(s + \frac{k-1}{2}\right) L(s, f) = \varepsilon_f \Lambda(1-s, f).$$

Definition

Let $H_k^\pm(N)$ be the set of $f \in H_k^*(N)$ with $\varepsilon_f = \pm 1$.

Definition (Test function)

$\phi : \mathbb{R}^n \rightarrow [0, \infty)$ is a test function if it is an even Schwartz function, $\hat{\phi}$ has compact support, and $\phi(0) > 0$.

Definition (1-level density)

Let $\rho_f = 1/2 + i\gamma_f$ be a non-trivial zero of $L(s, f)$. The **1-level density** of the $L(s, f)$ is

$$D(f; \phi) := \sum_{\gamma_f} \phi \left(\frac{\log R}{2\pi} \gamma_f \right),$$

where ϕ is a test function, R is the analytic conductor, and the sum is over normalized imaginary parts γ_f of zeros of the L -function associated with f .

Averaging over the family of $L(s, f)$

For a family $H_k^\pm(N)$ of cuspidal newforms weight k and squarefree level N , the first moment of the 1-level density is

$$\lim_{N \rightarrow \infty} \langle D(f; \phi) \rangle_\pm := \lim_{N \rightarrow \infty} \frac{1}{|H_k^\pm(N)|} \sum_{f \in H_k^\pm(N)} D(f; \phi).$$

To study n^{th} centered moments of $D(f; \phi)$ we explicitly compute

$$\lim_{N \rightarrow \infty} \langle (D(f; \phi) - \langle D(f; \phi) \rangle_\pm)^n \rangle_\pm$$

for $\hat{\phi}$ of large support.

We also study analogous random matrix theory quantities. For U a unitary $M \times M$ matrix with eigenvalues $e^{i\theta_n}$, define

Definition

For a Schwartz function ϕ on the real line, define

$$Z_\phi(U) := \sum_{n=1}^M \sum_{j=-\infty}^{\infty} \phi\left(\frac{M}{2\pi}(\theta_n + 2\pi j)\right).$$

Averaging over the family of matrices

For the analogous family of special orthogonal matrices, the n^{th} moment of $Z_\phi(U)$ averaged with Haar measure over $\text{SO}(\text{even})$ (or respectively $\text{SO}(\text{odd})$) is

$$\lim_{\substack{M \rightarrow \infty \\ M \text{ even/odd}}} \mathbb{E}_{\text{SO}(\text{even/odd})} [(Z_\phi(U) - \mu_\pm)^n]$$

where

$$\mu_\pm := \lim_{\substack{M \rightarrow \infty \\ M \text{ even/odd}}} \mathbb{E}_{\text{SO}(\text{even/odd})} [Z_\phi(U)].$$

Conjecture (Katz-Sarnak Density Conjecture)

The scaling limits of the distributions of zeros of families of automorphic L -functions agree with the scaling limits of eigenvalue distributions of classical subgroups of the unitary groups $U(N)$.

For our family of orthogonal L -functions, this posits

$$\begin{aligned} & \lim_{N \rightarrow \infty} \langle (D(f; \phi) - \langle D(f; \phi) \rangle_{\pm})^n \rangle_{\pm} \\ &= \lim_{\substack{M \rightarrow \infty \\ M \text{ even/odd}}} \mathbb{E}_{\text{SO}(\text{even/odd})} [(Z_{\phi}(U) - \mu_{\pm})^n] \end{aligned}$$

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Within $\text{supp}(\hat{\phi}) \subset (-\frac{1}{n}, \frac{1}{n})$, $\langle (D(f; \phi) - \langle D(f; \phi) \rangle_{\pm})^n \rangle_{\pm}$ look the same.

- By optimally picking ϕ , agreement of the Katz-Sarnak density conjecture allows us to lower bound what fraction of $L(s, f)$ have a fixed order of vanishing at $s = 1/2$.
- Larger support of $\hat{\phi}$ allows us to localize ϕ more and get better estimates near the central point.

Order of vanishing bounds for SO(odd)

Order vanishing	1-level	2-level	4 th centered moment*
5	0.222908	0.0674429	0.06580440
7	0.159220	0.0299746	0.00221997
9	0.123838	0.0168607	0.00036405

- These are upper bounds for vanishing at least r (number in order vanishing column).
- For the 1-level column, using the optimal 1-level bound from [ILS]. The support of the Fourier transform of the test function used is $(-2, 2)$.
- For the 2-level column, using the optimal 2-level bound from [BCDMZ]. The support of the Fourier transform of the test functions is $(-1, 1)$.
- For the 4th centered moment* column, the * signifies that we used the 4 copies of the naive test functions φ_{naive} . The support of the Fourier transform of the test function used is $(-1/3, 1/3)$.

Theorem (Iwaniec, Luo, Sarnak, 2000)

Let ϕ be an even Schwartz test function with support of $\hat{\phi}$ in $(-2, 2)$. Then for a family of cuspidal newforms of weight k and level N ,

$$\lim_{N \rightarrow \infty} \langle D(f; \phi) \rangle_+ = \lim_{\substack{M \rightarrow \infty \\ M \text{ even}}} \mathbb{E}_{\text{SO}(\text{even})} [Z_\phi(U)]$$

$$\lim_{N \rightarrow \infty} \langle D(f; \phi) \rangle_- = \lim_{\substack{M \rightarrow \infty \\ M \text{ odd}}} \mathbb{E}_{\text{SO}(\text{odd})} [Z_\phi(U)]$$

where $\text{SO}(\text{even})$ is the scaling limit of special orthogonal even matrices and $\text{SO}(\text{odd})$ is the scaling limit of special orthogonal odd matrices.

Theorem (Hughes, Miller, 2006)

Let $n \geq 2$, $\text{supp}(\hat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$. Then

$$\begin{aligned} & \lim_{\substack{N \rightarrow \infty \\ N \text{ prime}}} \langle (D(f; \phi) - \langle D(f; \phi) \rangle_{\pm})^n \rangle_{\pm} \\ = & \lim_{\substack{M \rightarrow \infty \\ M \text{ even/odd}}} \mathbb{E}_{(\text{SO}(\text{even/odd}))} [(Z_{\phi}(U) - \mu_{\pm})^n] \end{aligned}$$

Goal

For $L(s, f)$ and $a \leq n/2$, if $\text{supp}(\hat{\phi}) \subset (-\frac{1}{n-a}, \frac{1}{n-a})$, then the n^{th} centered moments of $D(f; \phi)$ averaged over $f \in H_k^{\pm}(N)$ satisfy the Density Conjecture.

We generalized many number-theoretic techniques to explicitly calculate the n^{th} centered moments of $D(f; \phi)$ for this support.

Hypothesis (Generalized Riemann Hypothesis, $L(s, \chi)$)

For every Dirichlet character χ , every nontrivial root s of $L(s, \chi)$ must have $\operatorname{Re}(s) = 1/2$.

Hypothesis (Generalized Riemann Hypothesis, $L(s, f)$)

For every cuspidal newform of weight k , level N , every nontrivial root s of $L(s, f)$ must have $\operatorname{Re}(s) = 1/2$.

To compute the n^{th} centered moments of $D(f; \phi)$, it suffices to convert the following sum over primes into a sum of integrals:

$$S_2^{(n)} := \sum_{p_1 \nmid N, \dots, p_n \nmid N} \prod_{j=1}^n \left(\hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{2 \log p_j}{\sqrt{p_j \log R}} \right) \langle \lambda_f(N) \prod_{j=1}^n \lambda_f(p_j) \rangle_*$$

where $\lambda_f(p_i)$ are the normalized Fourier coefficients of f .

- We explicitly calculate the term by applying the Petersson formula to convert Fourier coefficients into Bessel-Kloosterman terms.
- We remove negligible subterms before converting the rest into integrals.

Expand Fourier coefficients into exponential (*Bessel-Kloosterman*) sums

Expand exponential sums into terms involving Dirichlet characters and Gauss sums

Remove non-principal characters to get Ramanujan sums

Convert sums over integers into integrals of $\widehat{\phi}^k$

Subterms of $S_2^{(n)}$, where $\sum m_j \leq n, m_j \equiv n_j \pmod{2}$:

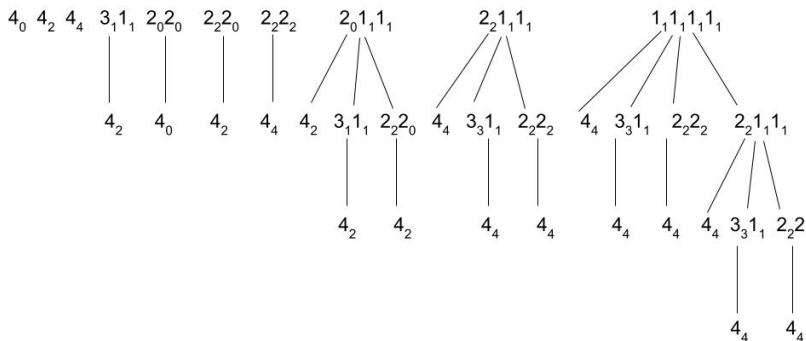
$$i^k \sqrt{N} \sum_{\substack{q_1 \nmid N, \dots, q_\ell \nmid N \\ q_j \text{ distinct}}} \prod_{j=1}^{\ell} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right)^{n_j} \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \right) \langle \lambda_f(N q_1^{m_1} \cdots q_\ell^{m_\ell}) \rangle$$

- As a increases more terms emerge and closed form for combinatorics in general case is difficult.
- We eliminate subterms of $S_2^{(n)}$ via combinatorics and the Weil bound.

Algorithm for Overcoming Combinatorial Obstructions

Here is an abstract representation of removing the distinctness condition for the fourth moment with unlimited support.

The main numeral represents how many primes are equal, and the subscript is the multiplicity of the prime in the Fourier coefficient



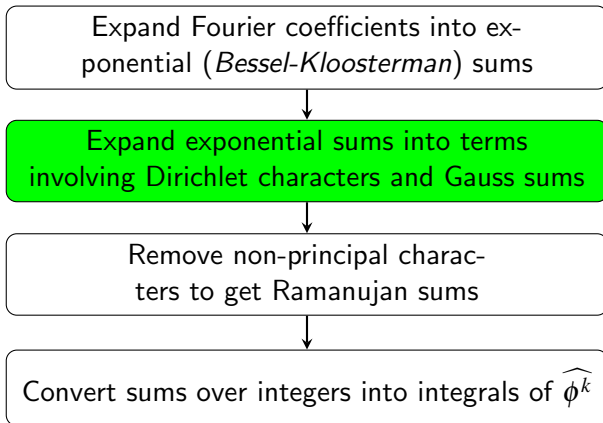
Example: Weil bound application

Lemma

Suppose $\text{supp}(\hat{\phi}) \subseteq \left(-\frac{1}{n-a}, \frac{1}{n-a}\right)$, and assume GRH for $L(s, f)$. If $\ell \leq n - a$, the above subterm is $O(N^{-\varepsilon})$.

Lemma

Suppose $\text{supp}(\hat{\phi}) \subseteq \left(-\frac{1}{n-a}, \frac{1}{n-a}\right)$, and assume GRH for $L(s, f)$. If for at least $\ell + a - n$ primes we have $n_i > m_i$, the above subterm is $O(N^{-\varepsilon})$.



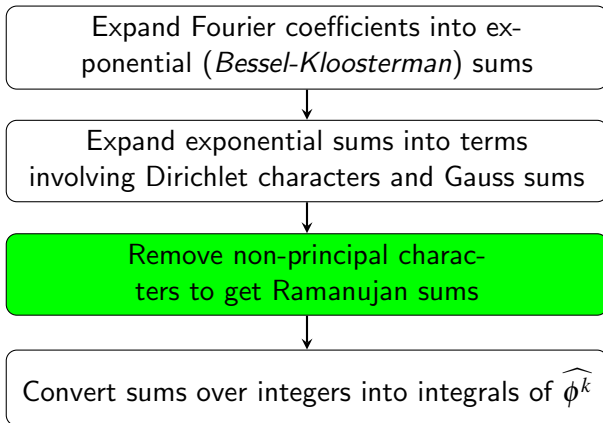
$$\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{q_1 \nmid N, \dots, q_\ell \nmid N} \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{b=1}^{\infty} \frac{S(m^2, Nq_1^{m_1} \dots q_\ell^{m_\ell}; Nb)}{b\sqrt{N}}$$

$$\times J_{k-1} \left(\frac{4\pi m \sqrt{q_1^{m_1} \dots q_\ell^{m_\ell}}}{b\sqrt{N}} \right) \prod_{j=1}^{\ell} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right)^{n_j} \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \right)$$

- Case on which primes among q_1, \dots, q_ℓ divide b
- Convert Kloosterman sum to Gauss sums over $\chi \pmod{b}$

$$S(m^2, NQ; Nb) = -\frac{1}{\varphi(b)} \sum_{\chi \pmod{b}} G_\chi(m^2) G_\chi((Q, b^\infty)) \bar{\chi} \left(\frac{Q}{(Q, b^\infty)} \right) \chi(N),$$

$$\text{where } G_\chi(n) := \sum_{a \pmod{b}} \chi(a) e(an/b).$$



- We show that non-principal characters in our sum do not contribute in the limit for support $(-2/n, 2/n)$. It turns out to be natural support for the random matrix theory equivalent.
- We need an application of GRH for Dirichlet L -functions:

$$\sum_{p \leq x} \chi(p) p^{it} \log p = O\left(x^{1/2} (bxt)^{\varepsilon}\right)$$

for non-principal character χ modulo b and any $\varepsilon > 0$.

- This result resolves an error in the work of Hughes and Miller, thus showing that the Katz-Sarnak Density Conjecture holds for $\text{supp}(\hat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$.

Via the inverse Mellin transform of the Bessel function and shifting the contour,

$$\begin{aligned}
 & \sum_{p_1, \dots, p_n < R^\sigma} J_{k-1} \left(\frac{4\pi m \sqrt{p_1 \dots p_n}}{b\sqrt{N}} \right) \prod_{j=1}^n \left(\hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\log p_j \bar{\chi}(p_j)}{\sqrt{p_j} \log R} \right) \\
 &= \frac{1}{2\pi i} \int_{\Re(s)=1} \left(\frac{4\pi m}{b\sqrt{N}} \right)^{-s} G_{k-1}(s) \left(\sum_{p < R^\sigma} \hat{\phi} \left(\frac{\log p}{\log R} \right) \frac{\log p \bar{\chi}(p)}{p^{(1+s)/2} \log R} \right)^n ds \\
 &\ll \frac{m}{b\sqrt{N}} R^{\sigma n/2+\varepsilon}. \quad (\text{apply previous character sum})
 \end{aligned}$$

Via the inverse Mellin transform of the Bessel function and shifting the contour,

$$\begin{aligned} & \sum_{p_1, \dots, p_n < R^\sigma} J_{k-1} \left(\frac{4\pi m \sqrt{p_1 \dots p_n}}{b\sqrt{N}} \right) \prod_{j=1}^n \left(\hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\log p_j \bar{\chi}(p_j)}{\sqrt{p_j} \log R} \right) \\ &= \frac{1}{2\pi i} \int_{\Re(s)=1} \left(\frac{4\pi m}{b\sqrt{N}} \right)^{-s} G_{k-1}(s) \left(\sum_{p < R^\sigma} \hat{\phi} \left(\frac{\log p}{\log R} \right) \frac{\log p \bar{\chi}(p)}{p^{(1+s)/2} \log R} \right)^n ds \\ &\ll \frac{m}{b\sqrt{N}} R^{\sigma n/2+\varepsilon}. \quad (\text{apply previous character sum}) \end{aligned}$$

So the entire Bessel-Kloosterman term is

$$\begin{aligned} &\ll N^{-1/2} \sum_{m \leq N^\varepsilon} \frac{1}{m} \sum_{\substack{(b,N)=1 \\ b < N^{2006}}} \frac{1}{b\varphi(b)} \sum_{\substack{\chi(b) \\ \chi \neq \chi_0}} |G_\chi(m^2) G_\chi(1)| \left(\frac{m}{b\sqrt{N}} R^{\sigma n/2+\varepsilon} \right) \\ &\ll N^{-1+\sigma n/2+\varepsilon}. \end{aligned}$$

Expand Fourier coefficients into exponential (*Bessel-Kloosterman*) sums



Expand exponential sums into terms involving Dirichlet characters and Gauss sums



Remove non-principal characters to get Ramanujan sums



Convert sums over integers into integrals of $\widehat{\phi^k}$

Under the Riemann Hypothesis for $\zeta(s)$, if $\text{supp}(\hat{\phi}) \subset (-\frac{1}{n-a}, \frac{1}{n-a})$, then, for $0 \leq \beta \leq a/2$, and for any N ,

$$\begin{aligned} & \sum_{v_1, \dots, v_{n-\beta}} \left[\prod_{i=1}^{n-\beta} \hat{\phi} \left(\frac{\log v_i}{\log R} \right) \left(\frac{\chi_0(v_i) \Lambda(v_i)}{\sqrt{v_i} \log R} \right) \right] J_{k-1} \left(\frac{4\pi m \sqrt{v_1 \dots v_{n-\beta}}}{b\sqrt{N}} \right) \\ &= \sum_{c=0}^{a-\beta-1} \sum_{j=c}^{a-\beta-1} (-1)^{j-c} \binom{n-\beta}{j} \binom{j}{c} \frac{b\sqrt{N}}{2\pi m \log R} \\ & \quad \times \sum_{r_1, \dots, r_c=1}^{\infty} \left[\prod_{i=1}^c \hat{\phi} \left(\frac{\log r_i}{\log R} \right) \frac{\chi_0(r_i) \Lambda(r_i)}{r_i \log R} \right] \\ & \quad \times \int_{x=0}^{\infty} J_{k-1}(x) \widehat{\phi^{n-\beta-c}} \left(\frac{2 \log(bx \sqrt{N / \prod_{i=1}^c r_i} / 4\pi m)}{\log R} \right) dx + O(N^{-\epsilon}). \end{aligned}$$

$$\begin{aligned}
 & \frac{2^{n+1}\pi}{\sqrt{N}} \sum_{p_1, \dots, p_n} \prod_{j=1}^n \hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\log p_j}{\sqrt{p_j} \log R} \langle \lambda_f(Np_1 \dots p_n) \rangle_* \\
 = & - \sum_{v=0}^{a-1} \binom{n}{v} \sum_{i=0}^{a-v-1} (-1)^i \binom{n-v}{i} \frac{2^n}{\log R} \sum_{p_1, \dots, p_v} \sum_{m \leq N^\epsilon} \frac{1}{m^2} \sum_{\substack{(b, N)=1 \\ b < N^2}} \frac{R(m^2, b) R(p_1 \dots p_v, b)}{\varphi(b)} \\
 & \times \int_{x=0}^{\infty} J_{k-1}(x) \widehat{\phi}^{n-v} \left(\frac{2 \log(bx\sqrt{N}/(4\pi m\sqrt{p_1 \dots p_v}))}{\log R} \right) dx \prod_{j=1}^v \hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\log p_j}{p_j \log R}.
 \end{aligned}$$

- Largest term within $S_2^{(n)}$ and also the simplest to work with.
- Need to simplify Ramanujan sums to show agreement with random matrix theory.

Theorem

For $\text{supp}(\hat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$, the Katz-Sarnak Density Conjecture holds.

- For support $\left(-\frac{1}{n-3}, \frac{1}{n-3}\right)$ and $\left(-\frac{1}{n-4}, \frac{1}{n-4}\right)$, we've completed the casework and converted sums to integrals, and have left to manipulate the integrals to look like the random matrix theory results.
- Showing agreement with random matrix theory for support larger than $\left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$ becomes significantly more difficult, as non-main terms begin to contribute.

- Our work fully generalized the number theoretic techniques needed to convert these sums over primes into sums over integers.
- We are currently generalizing the work of Iwaniec, Luo, and Sarnak to convert these terms to integrals in order to show agreement with random matrix theory.

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- 1 C. Hughes and S. Miller. "Low lying zeros of L-functions with orthogonal symmetry". In: *Duke Mathematical Journal* 136 (Aug. 2005). DOI: 10.1215/S0012-7094-07-13614-7.
- 2 H. Iwaniec, W. Luo, and P. Sarnak. "Low lying zeros of families of L-functions". In: *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* 91 (1999), pp. 55-131.
- 3 N. Katz and P. Sarnak. "Zeroes of zeta functions and symmetry". In: *Bulletin of the American Mathematical Society* 36 (1999), pp. 1-26.

Theorem (ILS)

Let ψ be an even Schwartz function with $\text{supp}(\widehat{\psi}) \subset (-2, 2)$. Then

$$\begin{aligned} & \sum_{m \leq N^\varepsilon} \frac{1}{m^2} \sum_{(b, N)=1} \frac{R(m^2, b)R(1, b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y) \widehat{\psi} \left(\frac{2 \log(by\sqrt{N}/4\pi m)}{\log R} \right) \frac{dy}{\log R} \\ &= -\frac{1}{2} \left[\int_{-\infty}^{\infty} \psi(x) \frac{\sin 2\pi x}{2\pi x} dx - \frac{1}{2} \psi(0) \right] + O \left(\frac{k \log \log kN}{\log kN} \right) \end{aligned}$$

where $R = k^2 N$ and φ is Euler's totient function.

Lemma

Assume GRH for $L(s, f)$. If $\text{supp}(\phi) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$ then

$$\lim_{\substack{N \rightarrow \infty \\ N \text{ prime}}} S_2^{(n)} = 2^{n-1} \left[\int_{-\infty}^{\infty} \phi(x)^n \frac{\sin 2\pi x}{2\pi x} dx - \frac{1}{2} \phi(0)^n \right] \\ - n2^{n-1} \left[\int_{-\infty}^{\infty} \hat{\phi}(x_2) \int_{-\infty}^{\infty} \phi(x_1)^{n-1} \frac{\sin(2\pi x_1(1+|x_2|))}{2\pi x_1} dx_1 dx_2 - \frac{1}{2} \phi(0)^n \right].$$