Upper Bounds for the Lowest First Zero in Families of Cuspidal Newforms

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Background

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) \;=\; \sum_{n=1}^{\infty}\frac{a_f(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

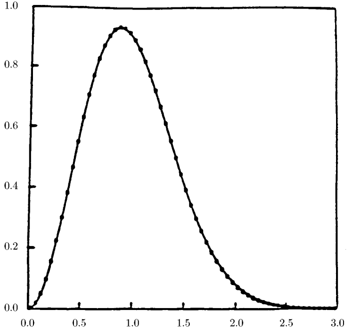
$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f)L(s,f) \;=\; \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko).

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

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Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)p^{-s}ds.$$

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Contour Integration (see Fourier Transform arising):

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)e^{-\sigma\log p}e^{-it\log p}ds.
$$

Knowledge of Zeros $\Leftrightarrow$ Knowledge of Coefficients.

Measures of Spacings: *n*-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbb{R}^{n-1}$.

n-level correlation

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1}-\alpha_{j_2},\ldots,\alpha_{j_{n-1}}-\alpha_{j_n}\right)\in B, j_i \neq j_k\right\}}{N}$$

(Instead of using a box, can use a smooth test function.)

Measures of Spacings: *n*-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbb{R}^{n-1}$.

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko).
2. 2 and 3-correlations of $\zeta(s)$ (Montgomery, Hejhal).
3. n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak).
4. n-level correlations for the classical compact groups (Katz-Sarnak).
5. Insensitive to any finite set of zeros.

Measures of Spacings: *n*-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

$$D_{n,f}(\phi) = \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f \gamma_f^{(j_1)}\right) \cdots \phi_n\left(L_f \gamma_f^{(j_n)}\right)$$

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1. Individual zeros contribute in limit.
2. Most of contribution is from low zeros.
3. Average over similar curves (family).

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Katz-Sarnak Conjecture

For a ‘nice’ family of *L*-functions, the *n*-level density depends only on a symmetry group attached to the family.

Normalization of Zeros

Local (hard, use C_f) vs Global (easier, use $\log C = |\mathcal{F}_N|^{-1} \sum_{f \in \mathcal{F}_N} \log C_f$). Hope: ϕ a good even test function with compact support, as $|\mathcal{F}| \to \infty$,

$$\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi) \quad = \quad \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\substack{j_1, \dots, j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i \left(\frac{\log C_f}{2\pi} \gamma_E^{(j_i)} \right)$$

$$\to \quad \int \cdots \int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x) dx.$$

Katz-Sarnak Conjecture

As $C_f \to \infty$ the behavior of zeros near $1/2$ agrees with $N \to \infty$ limit of eigenvalues of a classical compact group.

1-Level Densities

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,\mathrm{SO(even)}}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,\mathrm{SO}}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,\mathrm{SO(odd)}}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1 \end{cases}$$

Main Tools

1. Control of conductors: Usually monotone, gives scale to study low-lying zeros.
2. Explicit Formula: Relates sums over zeros to sums over primes.
3. Averaging Formulas: Petersson formula in Iwaniec-Luo-Sarnak, Orthogonality of characters in Fiorilli-Miller, Gao, Hughes-Rudnick, Levinson-Miller, Rubinstein.

Applications of *n*-level density

One application: bounding the order of vanishing at the central point.

$\text{Average rank} \cdot \phi(0) \ \leq \ \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.

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$\text{Average rank} \cdot \phi(0) \ \leq \ \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.

Can also use to bound the percentage that vanish to order r for any r.

Theorem (Miller, Hughes-Miller)

Using n-level arguments, for the family of cuspidal newforms of prime level $N \to \infty$ (split or not split by sign), for any r there is a c_r such that probability of at least r zeros at the central point is at most $c_n r^{-n}$.

Better results using 2-level than Iwaniec-Luo-Sarnak using the 1-level for $r \geq 5$.

n-Level Density / Moments for Cuspidal Newform: Cohen et. al.

Let $n \geq 2$ and $\mathrm{supp}(\phi) \subset (-\frac{2}{n}, \frac{2}{n})$. Define

$$\sigma_\phi^2 := 2\int_{-\infty}^{\infty} |y|\widehat{\phi}(y)^2 dy$$

$$R(m,i;\phi) := 2^{m-1}(-1)^{m+1}\sum_{l=0}^{i-1}(-1)^l \binom{m}{l} \left(-\frac{1}{2}\phi^m(0) + \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\widehat{\phi}(x_2)\cdots\widehat{\phi}(x_{l+1}) \int_{-\infty}^{\infty}\phi^{m-l}(x_1)\frac{\sin(2\pi x_1(1+|x_2|+\cdots+|x_{l+1}|))}{2\pi x_1}dx_1\cdots dx_{l+1}\right)$$

$$S(n,a,\phi) := \sum_{l=0}^{\lfloor\frac{a-1}{2}\rfloor}\frac{n!}{(n-2l)!l!}R(n-2l,a-2l,\phi)\left(\frac{\sigma_\phi^2}{2}\right)^l \text{ then}$$

$$\lim_{\text{prime } N\to\infty}\left\langle\left(D(f;\phi)-\langle D(f;\phi)\rangle_\pm\right)^n\right\rangle_\pm = (n-1)!!\sigma_\phi^n 1_{n \text{ even}} \pm S(n,a;\phi).$$

Results

Previous Results

Question

Assuming the GRH, how far up must we go on the critical line before we are assured that we will see the first zero?

Previous work mostly on first (lowest) zero of an L-function. Assume GRH, zeros of the form $\frac{1}{2} + i\gamma$.

Previous Results

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Assuming the GRH, how far up must we go on the critical line before we are assured that we will see the first zero?

Previous work mostly on first (lowest) zero of an *L*-function. Assume GRH, zeros of the form $\frac{1}{2} + i\gamma$.

- S. D. Miller: *L*-functions of real archimedian type has $\gamma < 14.13$.
- J. Bober, J. B. Conrey, D. W. Farmer, A. Fujii, S. Koutsoliotas, S. Lemurell, M. Rubinstein, H. Yoshida: General *L*-function has $\gamma < 22.661$.

Previous Results

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Previous work mostly on first (lowest) zero of an L-function. Assume GRH, zeros of the form $\frac{1}{2} + i\gamma$.

- J. Mestre: Elliptic curves: first zero occurs by $O(1/\log\log N_E)$, where N_E is the conductor (expect order $1/\log N_E$).

- J. Goes and S. J. Miller: One-Parameter Family of Elliptic Curves of rank r: $r + \frac{1}{2}$ normalized zeros on average within the band $\approx (-\frac{0.551329}{\sigma}, \frac{0.551329}{\sigma})$.

New Results: S. J. Miller and Tang

Theorem: Upper Bound Lowest First Zero in Even Cuspidal Families

For an odd $n = 2m+1$, whenever ω satisfies this following inequality

$$-\left(\widehat{\phi}_\omega(0) + \frac{1}{2}\int_{-\sigma/n}^{\sigma/n} \widehat{\phi}_\omega(y)dy\right)^n < 1_{n \text{ even}}(n-1)!!\sigma_{\phi_\omega}^n + S(n, a; \phi_\omega),$$

at least one form with at least one normalized zero in $(-\omega, \omega)$. Can take

$$\omega_{\min}(\sigma, h) > \left(-\frac{\sigma \int_0^1 h(u)^2\, du + \frac{\sigma^2}{4}\int_{-1}^1 \int_0^{2/\sigma} h(u)h(v-u)\, dv\, du}{\frac{1}{\sigma}\int_0^1 h(u)h''(u)\, du + \frac{1}{4}\int_{-1}^1 \int_0^{2/\sigma} h(u)h''(v-u)\, dv\, du}\right)^{-\frac{1}{2}} \frac{1}{\pi}.$$

Only know for $\sigma < 2$ (under GRH).
Get $\omega_{\min}(2, h) \approx 0.25$ for $h = \cos(\pi y/2)$.

New Results: S. J. Miller and Tang

Theorem: Normalized Zeros Near the Central Point

$P_{r,\rho}(\mathcal{F})$: percent of forms with at least r normalized zeros in $(-\rho, \rho)$.

For even n and $r \geq \mu(\phi, \mathcal{F})/\phi(\rho)$:

$$P_{r,\rho}(\mathcal{F}) \ \leq \ \frac{1_{n \text{ even}}(n-1)!!\sigma_\phi^n + S(n, a; \phi)}{(r\phi(\rho) - \mu(\phi, \mathcal{F}))^n}.$$

Explicit Bounds

Number of zeros	2-level	4-level	6-level
6	N/A	10.849910	48.154279
16	N/A	0.004235	$2.83230 \cdot 10^{-4}$
26	N/A	$3.541901 \cdot 10^{-4}$	$6.716802 \cdot 10^{-6}$
28	420.045063	$2.486819 \cdot 10^{-4}$	$3.943864 \cdot 10^{-6}$
30	20.991406	$1.796948 \cdot 10^{-4}$	$2.418466 \cdot 10^{-6}$
32	6.651738	$1.330555 \cdot 10^{-4}$	$1.538761 \cdot 10^{-6}$
34	3.220871	$1.006126 \cdot 10^{-4}$	$1.010576 \cdot 10^{-6}$

Table: Upper bound on probability of forms with at least r normalized zeros within 0.8 average spacing from central point, using naive test function with support $2/n$.
"N/A" means restriction in our theorem not met.

Constructions and Proofs

Preliminaries

- Convolution:

$$(A * B)(x) = \int_{-\infty}^{\infty} A(t)B(x-t)dt.$$

- Fourier Transform:

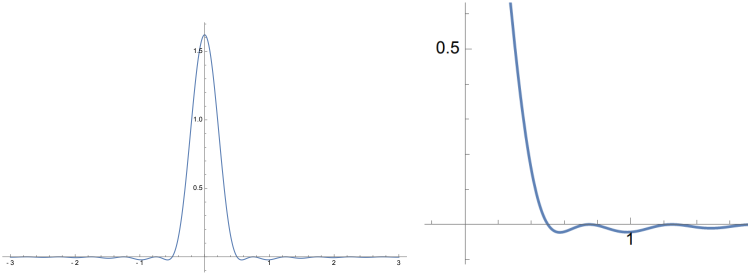
$$\widehat{A}(y) = \int_{-\infty}^{\infty} A(x)e^{-2\pi ixy}dx$$
$$\widehat{A''}(y) = -(2\pi y)^2\widehat{A}(y).$$

- Lemma: $\widehat{(A * B)}(y) = \widehat{A}(y) \cdot \widehat{B}(y)$;
  in particular, $\widehat{(A * A)}(y) = \widehat{A}(y)^2 \geq 0$ if A is even.

Construction of Test Function

Create compactly supported $\widehat{\phi}(y)$.

- Choose $h(y)$ even, twice continuously differentiable, supported on $(-1, 1)$, monotonically decreasing.
- $f(y) := h\left(\frac{2y}{\sigma/n}\right)$.
- $g(y) := (f * f)(y), \quad \widehat{g}(x) = \widehat{f}(x)^2 \geq 0.$
- $\widehat{\phi}_\omega(y) := g(y) + (2\pi\omega)^{-2} g''(y)$ thus $\phi_\omega(x) = \widehat{g}(x) \cdot (1 - (x/\omega)^2)$.



Plot of $\phi_\omega(x) = \widehat{g}(x) \cdot (1 - (x/\omega)^2)$, for $h = \cos\left(\frac{\pi y}{2}\right)$, $\sigma = 2$, $\omega = .5$, and $n = 1$.

Sketch of Proof: Key Expansion

Replace mean from finite N with the limit:

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \left(\sum_j \phi\left(\gamma_{f,j} c_n\right) - \mu(\phi, \mathcal{F}) \right)^n = 1_{n \text{ even}}(n-1)!!\sigma_\phi^n \pm S(n, a; \phi),$$

and main term of the mean of the 1-level density of $\mathcal{F}_N$ is

$$\mu(\phi, \mathcal{F}) \ := \ \widehat{\phi}(0) + \frac{1}{2}\int_{-1}^{1} \widehat{\phi}(y)dy.$$

Key Observation

$$\lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \left(\sum_j \phi(\widetilde{\gamma}_{f,j} c_n) - \mu(\phi, \mathcal{F}) \right)^n$$
$$= 1_{n \text{ even}}(n-1)!!\sigma_\phi^n \pm S(n, a; \phi).$$

$$\phi_\omega(x) = \widehat{g}(x) \cdot (1 - (x/\omega)^2).$$

- $\phi_\omega(x) \geq 0$ when $|x| \leq \omega$, and $\phi_\omega(x) \leq 0$ when $|x| > \omega$.
- Contribution of zeroes for $|x| \geq \omega$ is non-positive.
- As n odd, doesn't decrease if drop these non-positive contributions: why we restrict to odd n.

Sketch of Proof: Proof by Contradiction

Dropping negative contributions:

$$\lim_{\substack{N\to\infty \\ N\text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \left(\sum_{|\gamma_{f,j}|\le\omega} \phi_\omega(\gamma_{f,j} c_n) - \mu(\phi_\omega, \mathcal{F}) \right)^n \ \geq \ S(n, a; \phi_\omega).$$

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Assume no forms have a zero on the interval $(-\omega, \omega)$:

$$\lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} (-\mu(\phi_\omega, \mathcal{F}))^n \ \ge\ S(n, a; \phi_\omega),$$

$$(-\mu(\phi_\omega, \mathcal{F}))^n \lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} 1 \ \ge\ S(n, a; \phi_\omega).$$

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$$(-\mu(\phi_\omega, \mathcal{F}))^n \lim_{\substack{N\to\infty \\ N\text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} 1 \ \ge \ S(n, a; \phi_\omega).$$

As $\lim_{\substack{N\to\infty \\ N\text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} 1 = 1$, get

$$(-\mu(\phi_\omega, \mathcal{F}))^n \ \ge \ S(n, a; \phi_\omega).$$

Sketch of Proof: Continued

Because of the compact support of $\widehat{\phi}_\omega$,

$$-\left(\widehat{\phi}_\omega(0) + \frac{1}{2}\int_{-\sigma/n}^{\sigma/n} \widehat{\phi}_\omega(y)dy\right)^n \geq S(n, a; \phi_\omega).$$

Thus, if ω satisfies the following inequality

$$-\left(\widehat{\phi}_\omega(0) + \frac{1}{2}\int_{-\sigma/n}^{\sigma/n} \widehat{\phi}_\omega(y)dy\right)^n < S(n, a; \phi_\omega),$$

we get a contradiction, so at least one form has a normalized zero in $(-\omega, \omega)$.

Explicit Bound from 1-Level Density

First Zero from 1-Level

The first zero of the family of cuspidal newforms exists on the interval $(-\omega_{\min}, \omega_{\min})$, where

$$\omega_{\min}(\sigma, h) \;>\; \left(-\frac{\sigma \int_0^1 h(u)^2\, du + \frac{\sigma^2}{4} \int_{-1}^1 \int_0^{2/\sigma} h(u)h(v-u)\, dv\, du}{\frac{1}{\sigma} \int_0^1 h(u)h''(u)\, du + \frac{1}{4} \int_{-1}^1 \int_0^{2/\sigma} h(u)h''(v-u)\, dv\, du} \right)^{-\frac{1}{2}} \frac{1}{\pi}.$$

Number theory known only for $\sigma < 2$ (under GRH).

Get $\omega_{\min}(2, h) \approx 0.25$ for $h = \cos(\pi y/2)$.

Remarks on Computation and Support σ

- Restrictions with higher level computation.
- Riemann Sum approximation.
- Currently worse bounds with $\sigma = 2$ for larger *n*.
- Higher level yields better bounds if support large.
- Larger *n* better if σ larger.

Main Theorem 2

Theorem: Normalized Zeros Near the Central Point

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For even n and $r \geq \mu(\phi, \mathcal{F})/\phi(\rho)$:

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Naive Test Function

The naive test functions are the Fourier pair

$$\phi_{\text{naive}}(x) \ = \ \left(\frac{\sin(\pi v_n x)}{(\pi v_n x)}\right)^2 \quad , \quad \widehat{\phi}_{\text{naive}}(y) = \frac{1}{v_n}\left(y - \frac{|y|}{v_n}\right)$$

for $|y| \ < \ v_n$ where v_n is the support.

Sketch of Proof

Even n, dropping all with less than r zeros in $(-\rho, \rho)$ drops a non-negative sum:

$$\lim_{\substack{N\to\infty \\ N\text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_{N,r}^{(\rho)}} \left(\sum_{|\gamma_{f,j}|\le\rho} \phi(\gamma_{f,j}c_n) + T_f(\phi) - \mu(\phi,\mathcal{F}) \right)^n \le \dots.$$

Replace the summation of $\phi(\gamma_{f,j}c_n)$ with $r\phi(\rho)$; can drop $T_f(\phi)$ and not increase LHS if $r \ge \mu(\phi,\mathcal{F})/\phi(\rho)$:

$$\lim_{\substack{N\to\infty \\ N\text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_{N,r}^{(\rho)}} (r\phi(\rho) - \mu(\phi,\mathcal{F}))^n \le \dots.$$

$$P_{r,\rho}(\mathcal{F}) \le \frac{1_{n\text{ even}}(n-1)!!\sigma_\phi^n + S(n,a;\phi)}{(r\phi(\rho)-\mu(\phi,\mathcal{F}))^n}.$$

Explicit Bounds

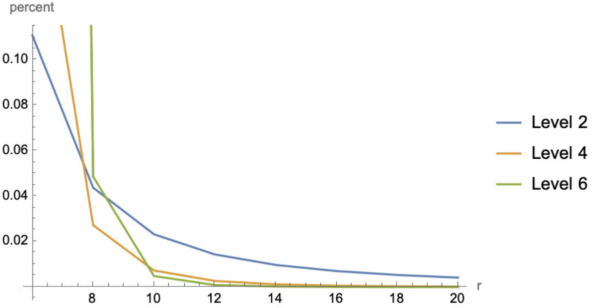

Figure: Percentage vs. number of zeros (for a fixed $\rho = .4$).

Higher levels starts above lower when r small, decrease faster and eventually gives better results as r grows.

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Number of zeros	2-level	4-level	6-level
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Future Works

Improving Bounds

- Optimize test function.
- Increase support of test function.
- Recent studies increased the support to 4 (Baluyot, Chandee, and Li) for a certain group of L-functions....

Reference

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