Identifying Symmetry Groups of Low-Lying Zeros of Families of L-Functions

Brown University

Number Theory and Random Matrix Theory Workshop (CMS Summer 2005)

Steven J. Miller (Joint with Eduardo Dueñez)

Waterloo, June 1st, 2005

http://www.math.brown.edu/~sjmiller
Measures of Spacings: $n$-Level Correlations

- $n$-Level correlations for the classical compact groups (Katz-Sarnak)
- $n$-Level correlations for all automorphic cuspidal $L$-fns (Rudnick-Sarnak)
- Pair and triple correlations of $\zeta(s)$ (Montgomery, Hejhal)
- Normalized spacings of $\zeta(s)$ starting at 1020 (Odlyzko)

**Results on Zeros (assuming GRH):**

\[
\lim_{N \to \infty} \frac{N}{\# \left\{ N \geq \lVert f \rVert \neq \lVert g \rVert, \exists B \in \left( \frac{u}{2}, \frac{u-1}{2}, \ldots, \frac{u-\nu-1}{2} \right) \right\}}
\]

box. Define the $n$-Level correlation by an increasing sequence of numbers $\{x_n\}$. 

universal to any finite set of zeros

insensitive to any finite set of zeros
Measures of Spacings: $n$-Level Density and Families

Let $g_i$ be even Schwartz functions whose Fourier Transform is compactly supported. Let $L(s, f)$ be an $L$-function with zeros at $\frac{1}{2} + i \gamma_f$ ($\gamma_f \in \mathbb{R}$) and conductor $C_f$. Define the $n$-level density by

$$D_{n, f}(g) = \sum_{j_1, \ldots, j_n} g_{j_1} \ldots g_{j_n} \frac{\log C_f}{2\pi} \left( \frac{\gamma_{f, j_1}}{2\pi} \right) \ldots \left( \frac{\gamma_{f, j_n}}{2\pi} \right)$$

Individual zeros contribute in limit

Most of contribution is from low zeros

Average over similar $L$-functions (family)

To any geometric family, Katz-Sarnak predict the $n$-level density depends only on a symmetry group (a classical compact group) attached to the family.
Conjecture: Distribution of Zeros near Central Point agrees with Distribution of Eigenvalues near 1 of a Classical Compact Group.

\[ n \rho(n)(x) \delta(u) \int \cdots \int \]

\[ x \rho(x)(x) \delta(u) \int \cdots \int \]

\[ \left( \frac{\nu}{\log \nu} \right)^{\nu \prod f} \prod \prod \frac{|N \mathcal{L}|}{1} = (b) f^{u} D \prod \frac{|N \mathcal{L}|}{1} \]

as \( \nu \rightarrow \infty \).
Let $G$ be one of the classical compact groups: Unitary, Symplectic, Orthogonal (or $SO(\text{even})$, $SO(\text{odd})$).

For test functions with $\text{supp} \delta \subset (0)$, 1-level density of

\[
\begin{cases}
\delta \; \text{is Orthogonal} & \delta = -1 \\
\delta \; \text{is Symplectic} & \delta = 1 \\
\delta \; \text{is Unitary} & \delta = 0
\end{cases}
\]

where

\[
\frac{\mathcal{Z}}{(0)\delta} \delta_c - (0)\delta
\]

is $\delta$.
Characters:

- Hughes-Rudnick, Miller: Families of Primitive Dirichlet

Orthogonal:

- Iwaniec-Luo-Sarnak: \( 1 \)-level density for \( H_k(N) \), \( N \text{ square-free} \).
- Miller, Young: Families of elliptic curves.
- Güloglu: \( 1 \)-level for \( f = \text{Sym}_r f \) at \( H_k(1) \), \( r \) even.

Symplectic:

- Rubinstein: \( n \)-level densities for \( L(s, \chi) \).
- Güloglu: \( 1 \)-level for \( f = \text{Sym}_r f \) at \( H_k(1) \), \( r \) even.

Unitary:

- Hughes-Rudnick, Miller: Families of primitive Dirichlet
- Young: Families of elliptic curves.

Free:

- Iwaniec-Luo-Sarnak: \( 1 \)-level density for \( H \), \( \mathbb{Z}^+ \) square-free.

Some Results: Simple Families
Identifying the Symmetry Groups

Often an analysis of the monodromy group in the function field suggests the answer. Otherwise, the family with odd signs, symplectic symmetry, or the family with even signs, orthogonal symmetry, is identified. Folklore Conjecture: If all signs are even and no corresponding L-function has sign changes, then the symplectic symmetry group is SO(2n), and otherwise, the orthogonal symmetry group is SO(2n+1).

How to identify symmetry groups in general? One possibility is by the signs of the functional equation:

\[ \text{Folklore Conjecture: If all signs are even and no corresponding L-function has sign changes, then the symplectic symmetry group is SO(2n), and otherwise, the orthogonal symmetry group is SO(2n+1).} \]

Tools: Explicit Formula, Orthogonality of Characters / Petersson Formula, L-functions. All simple families studied to date are built from GL_1 or GL_2 groups.
Some Results: Rankin-Selberg Convolution of Families

Notation:

\text{axed cuspidal automorphic representation of } GL_n \otimes \mathbb{Q};
\text{axed Hecke-Maass cuspform of level 1};
F_d: family of primitive quadratic characters;
F_{sym}^2 H_k: family of symmetric squares of above family;
\mathcal{J}_{P \chi}^{\text{sym}}(s, \phi) \text{ has a pole};
\mathcal{J}_{\text{sym}} F \times \phi \text{ is entire};
\mathcal{J}_{\text{sym}} F \times \phi \text{ has a pole};
\mathcal{J}_{P \chi} F \times \nu \text{ a fixed cuspidal automorphic representation of } \text{GL}_n \otimes \mathbb{Q}.

Orthogonal:

\text{Rubinstein: } F_d \text{ if } \mathcal{J}_{P \chi}^{\text{sym}}(s, \phi) \text{ has a pole}.
\text{Dueñez-Miller: } F_{sym}^2 H_k.

Symplectic:

\text{Rubinstein: } F_{sym}^2 H_k \text{ if } \mathcal{J}_{P \chi}^{\text{sym}}(s, \phi) \text{ has a pole}.
\text{Dueñez-Miller: } F_{sym}^2 H_k.

\text{Family of primitive quadratic characters, a fixed Hecke-Maass cusp form of level 1,}
Usually no contribution from $\Re \lambda > 1$ (Ramanujan Conjectures),

$\psi$ even Schwartz function, compactly supported.

$$\frac{\nu \Re \mathfrak{d} \Re \mathfrak{l}}{d \Re \mathfrak{l}} \frac{\zeta}{d \Re \mathfrak{l}} \left( \frac{\nu \Re \mathfrak{d} \Re \mathfrak{l}}{d \Re \mathfrak{l}} \lambda \right) \bigotimes_{n=1}^{d} \bigotimes_{\mathfrak{p} | n} \zeta - (0) \mathfrak{b} = \left( \frac{\nu \Re \mathfrak{d} \Re \mathfrak{l}}{d \Re \mathfrak{l}} \lambda \right) \bigotimes_{\mathfrak{p}} \mathfrak{b}$$

$\cdot (s - d(d)) \nu \chi - 1 \bigotimes_{u} d \bigotimes_{u} = \frac{s \nu \chi}{u} \bigotimes_{u} = (\nu, s) \mathcal{T}$

$\cdot \nu \chi \bigotimes_{u} = (d) \nu \chi \nu \chi \bigotimes_{u} \{ (d) \nu \chi \}$

By GRH the non-trivial zeros are

$\cdot (\nu, s) \mathcal{T}$

$\nu \Re \mathfrak{d} < \nu \mathfrak{d}$

 Explicit Formula
\[ \begin{bmatrix} \text{Orthogonal} \\ \text{Symplectic} \\ \text{Unitary} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \mathcal{F} c = (\zeta d f)^{\frac{\mathcal{L} \exists f}{\mathcal{L}}} \sum \frac{|\mathcal{L}|}{1} \]

The corresponding classical compact group is determined by

Except for families of elliptic curves with rank, first sum zero in all known cases (rank zero families).

\[ \begin{bmatrix} (\zeta d f)^{\frac{\mathcal{L} \exists f}{\mathcal{L}}} \sum \frac{|\mathcal{L}|}{1} \\ (d f)^{\frac{\mathcal{L} \exists f}{\mathcal{L}}} \sum \frac{|\mathcal{L}|}{1} \end{bmatrix} \begin{bmatrix} \frac{\mathcal{H} \delta_0 d}{d \delta_0} \\ \frac{\mathcal{H} \delta_0 d}{d \delta_0} \end{bmatrix} b \sum_2 z = (\mathcal{F})\mathcal{S} \]

\[ \begin{bmatrix} (\zeta d f) u, f \alpha + \cdots + (d f) \mathcal{I} f \alpha = (\zeta d f)^{\frac{\mathcal{L} \exists f}{\mathcal{L}}} \sum \frac{|\mathcal{L}|}{1} \end{bmatrix} \]

Assuming conductors constant in family, have to study Level Density for a Family.
for our applications.

Technical restriction: need \( f \) and \( g \) unrelated (i.e., \( b \) is not the \( \text{contragredient of } f \)), need also \( b \) and \( f \) unrelated as well.

\[
\cdot (d)^{b} \chi \cdot (d)^{f} \chi = \bigcup_{u} \bigcup_{u} \cdot (d)^{i} \cdot f \chi \bigcup_{u}
\]

\[
\cdot (d)^{b} \cdot (d)^{f} \chi = \bigcup_{u} \bigcup_{u} \cdot (d)^{i} \cdot f \chi \bigcup_{u}
\]

\[
\cdot (d)^{b} \cdot (d)^{f} \chi = (d)^{b \times f} \chi
\]

\[
\cdot w \bigcup_{u} \cdot (d)^{b} \chi \bigcup_{u} \cdot (d)^{f} \chi = \bigcup_{u} \bigcup_{u} \cdot (d)^{i} \cdot f \chi \bigcup_{u}
\]

\[
\cdot w \bigcup_{u} \cdot (d)^{b} \chi \bigcup_{u} \cdot (d)^{f} \chi = \bigcup_{u} \bigcup_{u} \cdot (d)^{i} \cdot f \chi \bigcup_{u}
\]

\[
(\cdot (d)^{b} \cdot (d)^{f} \chi)
\]

\[
\cdot (d)^{b} \cdot (d)^{f} \chi
\]

\[
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\]

\[
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\]

\[
\cdot (d)^{b} \cdot (d)^{f} \chi
\]

1-Level Density for Rankin-Selberg Convolution of Families

\[
\cdot (d)^{b} \cdot (d)^{f} \chi
\]

\[
\cdot (d)^{b} \cdot (d)^{f} \chi
\]

\[
\cdot (d)^{b} \cdot (d)^{f} \chi
\]
To analyze $\mathcal{S}_{(\mathcal{G} \times \mathcal{F})^\Lambda}$, the symmetry types of the convolution of the families (cont)

To analyze $\mathcal{S}_{(\mathcal{G} \times \mathcal{F})^\Lambda}$, we must study

If one of the families is rank zero, so is $\mathcal{G}$, and hence $\mathcal{G}$ will not contribute.

If each family is of rank $0$, the symmetry type of the convolution is the product of the symmetry types.

If one of the families is rank zero, so is $\mathcal{G}$, and hence $\mathcal{G}$ will not contribute.
One-parameter families of elliptic curves over \( \mathbb{Q} \):

\[
\begin{align*}
E \colon y^2 &= x^3 + A(T)x + B(T) \text{ of rank } r \text{ over } \mathbb{Q}(T).
\end{align*}
\]

Surfaces with non-constant \((\mathcal{L})\)

\[
\eta^+ = \log \left( (d)^{\eta} \right) \sum_{\text{d mod } t} \frac{d}{I}
\]

Rational Elliptic Surfaces (Rosen and Silverman): If rank

\[
\cdot (d)^{\eta} = (d)^{d \omega + \eta} \quad d^\omega \text{ is of size } d^\omega (d)^{\eta} = (d)^{\eta}
\]

One-parameter family of elliptic curves over \((\mathcal{L})\)

\[
(\mathcal{L}) \mathcal{B} + x(\mathcal{L}) \mathcal{V} + \zeta x = \zeta h : \mathcal{Z}
\]

One-parameter family of elliptic curves over \((\mathcal{L})\)
- Level Density of One-Parameter Families of Elliptic Curves

Rescale zeros by average log-conductor (for convenience).

First sum contributes

Rescale zeros by average log-conductor of Elliptic Curves

$1$-Level Density of One-Parameter Families of Elliptic Curves
Second sum handled similarly:

Second Level Density of One-Parameter Families of Elliptic Curves

1-Level Density of One-Parameter Families of Elliptic Curves
Combining we find that the $1$-level density of a one-parameter family of elliptic curves over $\mathbb{Q}(T)$ is:

$$
\frac{(0)\delta^2}{(0)\delta} + \frac{\tau}{(0)\delta} + (0)\delta
$$

What is the symmetry type of one-parameter fam-

$\begin{pmatrix}
\text{(odd)} & \times & \text{I} \\
\text{SO} & & \\
\end{pmatrix}$

or

$\begin{pmatrix}
\text{(even)} & \times & \text{I} \\
\text{SO} & & \\
\end{pmatrix}$

Agrees with scaling limits of

If $E$ is the family of elliptic curves of rank $r$ over $\mathbb{Q}$. If $E_{r_1}$, $E_{r_2}$, $E_{r_i}$ is a one-parameter family of elliptic curves of rank $r_i$ over $\mathbb{Q}(T)$ with $i=1,2,3$. Then the level density of one-parameter families of elliptic curves is:

$$
\lim_{T \to \infty} \frac{1}{T} \sum_{\text{Gal}(L/\mathbb{Q})} \text{dim} \text{Hom}(\Lambda, E_{T, \ell})
$$

$\ell = \text{the level}$

Where

$\Lambda = \varphi \left( \frac{1}{2} \right)$
Similar to Goldfeld's results on quadratic twists of a fixed elliptic curve:

Ranks of families play no role in new family.

\[ \mathcal{D} \times \mathcal{D} \] is the density for \( \mathcal{D} \times \mathcal{D} \) at level 1.
Twisting by a Fixed Form

Consider a GL₂ form $L(s,f)$ with Satake parameters $\alpha$, of size $L$. Orthogonal

Thus

\[
\begin{align*}
\left( z d \right)^y \chi &- \left( z d \right)^y \chi (d) f \gamma = \\
\left( z d \right)^y \chi \cdot \left( z d \right)^y \chi & = \left( z d \right)^y \times f \chi
\end{align*}
\]

Thus

\[
\begin{align*}
\left[ I - \left( d \right)^y \chi (d) f \chi \right] & = \\
\left[ I - \left( d \right)^y \gamma \chi \right] & = \\
\left[ I - \gamma \chi + I + \gamma \chi \right] & = \\
\gamma \chi + \gamma \chi & = \left( z d \right)^y \chi
\end{align*}
\]

\[
\begin{align*}
I & = \alpha \chi + \gamma \chi = \left( d \right)^y \chi
\end{align*}
\]

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\left[ I - \left( d \right)^y \gamma \chi \right] & = \\
\left[ I - \gamma \chi + I + \gamma \chi \right] & = \\
\gamma \chi + \gamma \chi & = \left( z d \right)^y \chi
\end{align*}
\]

\[
\begin{align*}
I & = \alpha \chi + \gamma \chi = \left( d \right)^y \chi
\end{align*}
\]
If \( \chi \) is a quadratic character then for almost all primes, \( d \):

\[
(p \chi)_{\chi} \quad \text{if } d \text{ is entire,}
\]
\[
\frac{1}{d} \sum_{\nu \leq d} \nu \quad \text{if } d \text{ has a pole}
\]

\[
= \left( \frac{d}{\zeta} \right)^p \chi
\]

This is a cuspidal automorphic representation on \( GL_n \).

Twisting by a Fixed Form
Theory of Low-Lying Zeros is more than a theory of signs.

Summary
By the Birch and Swinnerton-Dyer conjecture and the Silverman specialization theorem, in a one-parameter family of elliptic curves of rank $r$ over $\mathbb{Q}(T)$, eventually all curves have rank at least $r$.

**DEFINITIONS**

\[
D_n; F_N(t) = \prod_{j=1}^{n} \log C_t^{\gamma_j}\n\]

**ASSUMPTIONS**

1. Parameter family of Ell Curves, rank $r$ over $\mathbb{Q}$, rational surface. Assume

   1. $\mathbb{Q}(T)$ non-constant;
   2. $\mathbb{Q}(T)$ has irreducible factor of degree $d$.

**Pass to positive parameter sub-seq where conductors polynomial of degree $m$.**

\[
\text{Pass to positive parameter sub-seq where conductors polynomial of degree $m$.}
\]

**FAMILY ZEROS OF ELLIPTIC CURVES WRT THE REMAINING ZEROS**

Appendix I: Numerical Data on the Independence of the
Theorem (M–):

Under previous conditions, as $N \to \infty$, $n = 1, 2$:

$$D(r) \to N \cdot \mathcal{Z}(x) W G(x)$$

where

$$\begin{cases} 
\text{SO} & \text{if all odd} \\
\text{SO}^{(\text{even})} & \text{if all even} \\
\text{SO} & \text{if half odd}
\end{cases} = 5$$

Dependence on $F$ through lower order correction terms.

Agree with Independent Model, note universality.

Divergent predictions for small support.

1 and 2-level densities confirm Katz-Sarnak, Birch and Swinnerton-Dyer model and Swinnerton-Dyer

Theorem (M–) (under previous conditions, as $N \to \infty$, $n = 1, 2$):

**Main Result**
Theoretical Distribution of First Normalized Zero

First normalized eigenvalue: 322,560 from $SO(7)$ with Haar Measure

First normalized eigenvalue: 230,400 from $SO(6)$ with Haar Measure
750 curves, $\log(\text{cond}) \in [9.2, 14.9]$; mean = .88

750 curves, $\log(\text{cond}) \in [12.2, 14.6]$; mean = 1.04

Rank 0 Curves: 1st Normalized Zero (Far left and right bins just for formatting)
\[
\text{Rank 2 Curves: 1st}\text{Normalized Zero}
\]

665 curves, \( \log(\text{cond}) \in [16, 16.5]; \text{mean} = 1.82 \)

665 curves, \( \log(\text{cond}) \in [10, 10.3125]; \text{mean} = 2.30 \)

\text{Ranks: 1st Normalized Zero}
$y^2 = x^3 + T^2 x + T^2$; $\text{1st Normalized Zero}$

$34$ curves, $\log(\text{cond}) \in [16.2, 23.3]; \text{mean} = 20.0$

$35$ curves, $\log(\text{cond}) \in [7.8, 16.1]; \text{mean} = 22.4$

Rank 2 Curves: $L + x L - x^3 = \frac{f}{\zeta}$
Bibliography


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