

Low-lying zeros of cuspidal Maass forms

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Introduction
From Random Matrix Theory to L-Function

Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at t_1, t_2, t_3, \dots

Question: What rules govern the spacings between the t_i ?

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Examples:

- Spacings b/w Energy Levels of Nuclei.
- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
- Spacings b/w Zeros of L -functions.

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p , define

$$\text{Prob}(A) = \prod_{1 \leq i < j \leq N} p(a_{ij}).$$

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Fix p , define

$$\text{Prob}(A) = \prod_{1 \leq i \leq j \leq N} p(a_{ij}).$$

- Want to understand eigenvalues of A .
- Eigenvalues will be real since matrix is symmetric.

Eigenvalue Distribution

To each A , attach a probability measure using **scaled** eigenvalues:

$$\mu_{A,N}(X) = \frac{1}{N} \sum_{i=1}^N \delta \left(x - \frac{\lambda_i(A)}{2\sqrt{N}} \right)$$

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Let A be an $N \times N$ matrix with eigenvalues $\lambda_j(A)$. Then

$$\text{Trace}(A^k) = \sum_{n=1}^N \lambda_j(A)^k,$$

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- We need the average k^{th} moment for typical A .
- This will give us information about the distribution of eigenvalues for typical A .

L-functions

Riemann zeta function:

$$\zeta(s) = \sum_n \frac{1}{n^s}$$

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Generalized Riemann Hypothesis (RH):

All non-trivial zeros have $\operatorname{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

RMT vs L -functions

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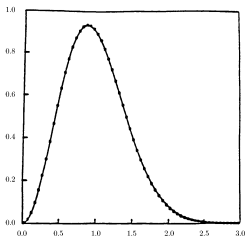
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Main idea: Eigenvalues \iff Zeros

So can use RMT to make conjectures about L -functions zeros.

Example: Spacings b/w zeros same as b/w eigenvalues of Complex Hermitian matrices.



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the 10^{20} th zero (from Odlyzko)

Sketch of proofs

Study L-functions like studied random matrix theory. Steps:

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Study L-functions like studied random matrix theory. Steps:

- 1 Determine correct scale for events (like in RMT...)
- 2 Develop an explicit formula relating what we want to study to something we understand (like in RMT...)
- 3 Use an averaging formula to analyze the quantities above (like in RMT...)

n-Level Density

Measures of Spacings: n -Level Density

n -level density for one function

$$D_{n,f}(\phi) = \sum_{\substack{j_1, \dots, j_n \\ \text{distinct}}} \phi_1(L_f \gamma_f^{(j_1)}) \cdots \phi_n(L_f \gamma_f^{(j_n)})$$

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- Zeros scaled by L_f
- Most of contribution is from low zeros.

Katz-Sarnak Conjecture

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For a 'nice' family of L -functions, the average n -level density depends only on a symmetry group attached to the family.

As $|\mathcal{F}| \rightarrow \infty$,

$$\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi) \rightarrow \int \cdots \int \phi(x) W_{n,g(\mathcal{F})}(x) dx.$$

1-level density

The Fourier Transforms for the 1-level densities are

$$\widehat{W}_{1,SO(\text{even})}(u) = \delta_0(u) + \frac{1}{2}\eta(u)$$

$$\widehat{W}_{1,SO}(u) = \delta_0(u) + \frac{1}{2}$$

$$\widehat{W}_{1,SO(\text{odd})}(u) = \delta_0(u) - \frac{1}{2}\eta(u) + 1$$

$$\widehat{W}_{1,Sp}(u) = \delta_0(u) - \frac{1}{2}\eta(u)$$

$$\widehat{W}_{1,U}(u) = \delta_0(u)$$

where $\delta_0(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1 \end{cases}$$

Cuspidal Maass Forms

Maass Forms

Definition: Maass Forms

A Maass form on a group $\Gamma \subset PSL(2, \mathbb{R})$ is a function $f : \mathcal{H} \rightarrow \mathbb{R}$ which satisfies:

- 1 $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$,
- 2 f vanishes at the cusps of Γ , and
- 3 $\Delta f = \lambda f$ for some $\lambda > 0$, where

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is the Laplace-Beltrami operator on \mathcal{H} .

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- Test Katz-Sarnak conjecture.

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is the Laplace-Beltrami operator on \mathcal{H} .

- Test Katz-Sarnak conjecture.
- Coefficients contain information about partitions.

L-function associated to Maass forms

Let u_j be a Maass form and write the Fourier expansion of u_j as

$$u_j(z) = \cosh(t_j) \sum_{n \neq 0} \sqrt{y} \lambda_j(n) K_{it_j}(2\pi|n|y) e^{2\pi inx}.$$

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Then we define the L -function attached to u_j as:

$$\begin{aligned} L(s, u_j) &= \sum_{n \geq 1} \frac{\lambda_j(n)}{n^s} \\ &= \prod_p \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_j(p)}{p^s}\right)^{-1} \end{aligned}$$

where

$$\alpha_j(p) + \beta_j(p) = \lambda_j(p), \quad \alpha_j(p)\beta_j(p) = 1.$$

Differences between L-functions

Dirichlet L-functions are in the GL_1 family:

$$L(s) = \prod_p \left(1 - \frac{\chi(p)}{p^s} \right)^{-1}$$

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Maass/holomorphic forms L-functions are GL_2 :

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n-level over a family

- Recall for Katz-Sarnak Conjecture, need to average *n*-level density over a family and take the limit of this parameter.

$$\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\substack{j_1, \dots, j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i \left(L_f \gamma_E^{(j_i)} \right)$$

$$\rightarrow \int \cdots \int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x) dx.$$

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$$\rightarrow \int \cdots \int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x) dx.$$

- For Dirichlet/cuspidal newform L-functions, there are many with a given conductor.
- Problem:** For Maass forms, expect at most one with a given conductor.

n-level over a family, continued

- **Solution:** Average over Laplace eigenvalues $\lambda_f = 1/4 + t_j^2$.

n-level over a family, continued

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- Use smooth weight function h ,

$$h_T(t_j) = e^{-(t_j - T)^2 / L^2},$$

which picks out eigenvalues in the window $[T - L, T + L]$.

n-level over a family, continued

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- Average 1-level density becomes

$$\begin{aligned} & \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} D_{n, u_j}(\phi) \\ &= \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \sum_{\substack{j_1, \dots, j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i \left(\frac{\gamma}{2\pi} \log R \right) \end{aligned}$$

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1-level density for one function

$$D(u_j; \phi) = \sum_{\gamma} \phi\left(\frac{\gamma}{2\pi} \log R\right)$$

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1-level density for one function

$$\begin{aligned}
 & D(u_j; \phi) \\
 &= \text{Terms involving } \Gamma + \frac{2}{\log R} \sum_p \frac{\log p}{p} \hat{\phi} \left(\frac{2 \log p}{\log R} \right) \\
 & - \sum_p \frac{2\lambda_j(p) \log p}{p^{\frac{1}{2}} \log R} \hat{\phi} \left(\frac{\log p}{\log R} \right) - \sum_p \frac{2\lambda_j(p^2) \log p}{p \log R} \hat{\phi} \left(\frac{2 \log p}{\log R} \right)
 \end{aligned}$$

- 1 Complex Analysis (contour shifts) \Rightarrow explicit formula to relate sums over zeros to sums over primes.

1-Level Density

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$$\begin{aligned}
 & D(u_j; \phi) \\
 &= \hat{\phi}(0) \frac{\log(1 + t_j^2)}{\log R} + \frac{2}{\log R} \sum_p \frac{\log p}{p} \hat{\phi}\left(\frac{2 \log p}{\log R}\right) \\
 &\quad - \sum_p \frac{2\lambda_j(p) \log p}{p^{\frac{1}{2}} \log R} \hat{\phi}\left(\frac{\log p}{\log R}\right) - \sum_p \frac{2\lambda_j(p^2) \log p}{p \log R} \hat{\phi}\left(\frac{2 \log p}{\log R}\right)
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- 2 Gamma function identities
- 3 Prime Number Theorem

Average 1-level density

The average 1-level density becomes:

$$\begin{aligned}
 & \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_t(t_j)}{\|u_j\|^2} D(u_j; \phi) \\
 &= \frac{\phi(0)}{2} + O\left(\frac{\log \log R}{\log R}\right) + \frac{1}{\sum_j \frac{h_t(t_j)}{\|u_j\|^2}} \sum_j \frac{h_t(t_j)}{\|u_j\|^2} \hat{\phi}(0) \frac{\log(1+t_j^2)}{\log R} \\
 & - \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{2 \log p}{p \log R} \hat{\phi}\left(\frac{\log p}{\log R}\right) \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \lambda_j(p) \\
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Kuznetsov Trace Formula

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To tackle terms with $\lambda_j(p)$ and $\lambda_j(p^2)$ we need the Kuznetsov Trace Formula:

$$\sum_j \frac{h(t_j)}{\|u_j\|^2} \lambda_j(m) \overline{\lambda_j(n)}$$

= some function that depends just on h, m, and n

Kuznetsov Trace Formula

$$\sum_j \frac{h(t_j)}{\|u_j\|^2} \lambda_j(m) \overline{\lambda_j(n)} + \frac{1}{4\pi} \int_{\mathbb{R}} \overline{\tau(m, r)} \tau(n, r) \frac{h(r)}{\cosh(\pi r)} dr =$$

$$\frac{\delta_{n,m}}{\pi^2} \int_{\mathbb{R}} r \tanh(r) h(r) dr + \frac{2i}{\pi} \sum_{c \geq 1} \frac{S(n, m; c)}{c} \int_{\mathbb{R}} J_{ir} \left(\frac{4\pi\sqrt{mn}}{c} \right) \frac{h(r)r}{\cosh(\pi r)} dr$$

where

$$\tau(m, r) = \pi^{\frac{1}{2} + ir} \Gamma(1/2 + ir)^{-1} \zeta(1 + 2ir)^{-1} n^{-\frac{1}{2}} \sum_{ab=|m|} \left(\frac{a}{b}\right)^{ir}.$$

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$$S(n, m; c) = \sum_{0 \leq x \leq c-1, \gcd(x, c)=1} e^{2\pi i(nx + mx^*)/c}$$

Kuznetsov Trace Formula

$$\sum_j \frac{h(t_j)}{\|u_j\|^2} \lambda_j(m) \overline{\lambda_j(n)} + \frac{1}{4\pi} \int_{\mathbb{R}} \overline{\tau(m, r)} \tau(n, r) \frac{h(r)}{\cosh(\pi r)} dr =$$

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- The only $\lambda(m) \overline{\lambda(n)}$ term that contributes is when $m = n = 1$.
- The $m = 1, n = p$ and $m = 1, n = p^2$ terms do not contribute because of the $\delta_{m,n}$ function.

Result: 1-level density

Theorem (AILZ, 2011)

Then if $T \rightarrow \infty$, the average 1-level density is

$$\begin{aligned} & \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} D(u_j; \phi) \\ &= \frac{\phi(0)}{2} + \hat{\phi}(0) + O\left(\frac{\log \log R}{\log R}\right) + O(T^{-\epsilon}) \end{aligned}$$

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- This matches with the **orthogonal family** density as predicted by Katz-Sarnak.
- Small support due to bounds on non-contributing terms in Kuznetsov formula.

2- Level Density

To differentiate between even and odd in orthogonal family, we calculated the 2-level density:

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 &= \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \prod_{i=1}^2 \left[\frac{\phi_i(0)}{2} + \hat{\phi}_i(0) \frac{\log(1+t_j^2)}{\log R} + O\left(\frac{\log \log R}{\log R}\right) \right. \\
 &\quad \left. - \sum_p \frac{2\lambda_j(p) \log p}{p^{\frac{1}{2}} \log R} \hat{\phi}_i\left(\frac{\log p}{\log R}\right) - \sum_p \frac{2\lambda_j(p^2) \log p}{p \log R} \hat{\phi}_i\left(\frac{2 \log p}{\log R}\right) \right]
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 \end{aligned}$$

Notice that will have **25 terms!**

To handle λ 's, we use Kuznetsov again.

Result: 2-level density

Theorem (AILZ, 2011)

$$\begin{aligned}
 D_{2,\mathcal{F}}^* &= \prod_{i=1}^2 \left[\frac{\phi_i(0)}{2} + \hat{\phi}_i(0) \right] + 2 \int_{-\infty}^{\infty} |z| \hat{\phi}_1(z) \hat{\phi}_2(z) dz \\
 &\quad - \phi_1(0)\phi_1(0) - 2\phi_1\hat{\phi}_2(0) + (\phi_1\phi_2)(0)\mathcal{N}(-1) \\
 &\quad + O\left(\frac{\log \log R}{\log R}\right)
 \end{aligned}$$

for $\sigma < 1/12$.

Note that $\mathcal{N}(-1)$ is a weighted percent that have odd sign in functional equation.

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Thank you!