Distribution of Eigenvalues of Weighted, Structured Matrix Ensembles

Olivia Beckwith[1], Steven J. Miller[2], and Karen Shen[3]

[1]Harvey Mudd College [2]Williams College [3]Stanford University

Young Mathematicians Conference 2011
Ohio State University

What is a random matrix?

What is a random matrix?

A matrix where the entries are chosen randomly according to some probability distribution p, i.e.:

What is a random matrix?

A matrix where the entries are chosen randomly according to some probability distribution p, i.e.:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix}$$

What is a random matrix?

A matrix where the entries are chosen randomly according to some probability distribution p, i.e.:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix}$$

$$\mathbb{P}(A) = \prod_{1 \le i,j \le N} p\left(a_{ij}\right)$$

What is a random matrix?

A matrix where the entries are chosen randomly according to some probability distribution p, i.e.:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix}$$

$$\mathbb{P}(A) = \prod_{1 \leq i,j \leq N} p(a_{ij})$$

Generally, we normalize p so that:

What is a random matrix?

A matrix where the entries are chosen randomly according to some probability distribution p, i.e.:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix}$$

$$\mathbb{P}(A) = \prod_{1 \le i,j \le N} p(a_{ij})$$

Generally, we normalize p so that:

$$\mathbb{E}(a_{ij}) = 0 \text{ and } Var(a_{ij}) = 1$$

Limiting Spectral Measure

Interested in the distribution of eigenvalues of A as $N \to \infty$

Limiting Spectral Measure

Interested in the distribution of eigenvalues of A as $N \to \infty$

Applications:

Limiting Spectral Measure

Interested in the distribution of eigenvalues of A as $N \to \infty$

Applications:

- Nuclear Physics

Limiting Spectral Measure

Interested in the distribution of eigenvalues of A as $N \to \infty$

Applications:

- Nuclear Physics
- Number Theory

Random Matrix Ensembles

Question: What happens when we impose structure on the entries of a matrix?

Random Matrix Ensembles

Question: What happens when we impose structure on the entries of a matrix?

To answer, study "families" or "ensembles" of random matrices:

Random Matrix Ensembles

Question: What happens when we impose structure on the entries of a matrix?

To answer, study "families" or "ensembles" of random matrices:

- Real Symmetric

Random Matrix Ensembles

Question: What happens when we impose structure on the entries of a matrix?

To answer, study "families" or "ensembles" of random matrices:

- Real Symmetric
- Real Symmetric Toeplitz:

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N-1} \\ b_1 & b_0 & b_1 & \cdots & b_{N-2} \\ b_2 & b_1 & b_0 & \cdots & b_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{N-1} & b_{N-2} & b_{N-3} & \cdots & b_0 \end{pmatrix} a_{ij} = b_{|i-j|}$$

Random Matrix Ensembles

Question: What happens when we impose structure on the entries of a matrix?

To answer, study "families" or "ensembles" of random matrices:

- Real Symmetric
- Real Symmetric Toeplitz:

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N-1} \\ b_1 & b_0 & b_1 & \cdots & b_{N-2} \\ b_2 & b_1 & b_0 & \cdots & b_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{N-1} & b_{N-2} & b_{N-3} & \cdots & b_0 \end{pmatrix} a_{ij} = b_{|i-j|}$$

- Real Symmetric Palindromic Toeplitz

Random Matrix Ensembles

Question: What happens when we impose structure on the entries of a matrix?

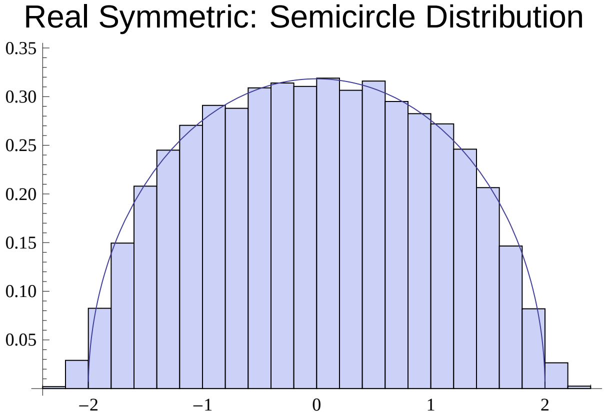
To answer, study "families" or "ensembles" of random matrices:

- Real Symmetric
- Real Symmetric Toeplitz:

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N-1} \\ b_1 & b_0 & b_1 & \cdots & b_{N-2} \\ b_2 & b_1 & b_0 & \cdots & b_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{N-1} & b_{N-2} & b_{N-3} & \cdots & b_0 \end{pmatrix} \quad a_{ij} = b_{|i-j|}$$

- Real Symmetric Palindromic Toeplitz
- etc.

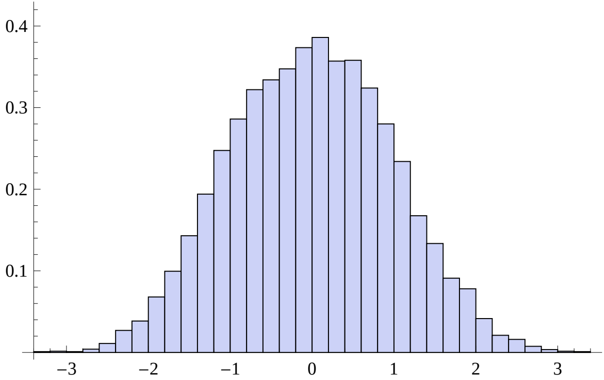
Previous Work



[Wigner, 1955]

Previous Work

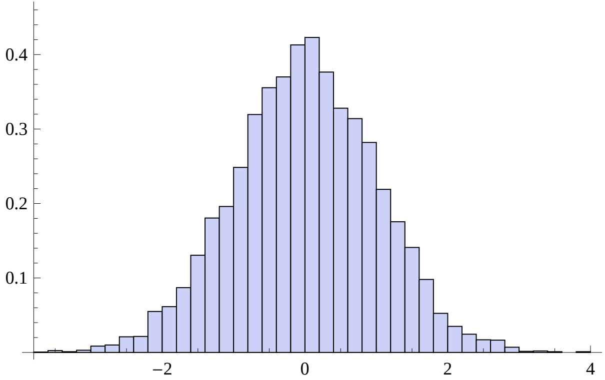
Real Symmetric Toeplitz: Almost Gaussian



[Hammond and Miller, 2005]

Previous Work

Real Symmetric Palindromic Toeplitz: Gaussian



[Miller, Massey and Sinsheimer, 2007]

Our Ensemble: Signed Toeplitz and Palindromic Toeplitz Matrices

For each entry, assign a randomly chosen $\epsilon_{ij} = \{1, -1\}$ such that $\epsilon_{ij} = \epsilon_{ji}$.

Our Ensemble: Signed Toeplitz and Palindromic Toeplitz Matrices

For each entry, assign a randomly chosen $\epsilon_{ij} = \{1, -1\}$ such that $\epsilon_{ij} = \epsilon_{ji}$.

Let $p = \mathbb{P}(\epsilon_{ij} = 1)$.

Our Ensemble: Signed Toeplitz and Palindromic Toeplitz Matrices

For each entry, assign a randomly chosen $\epsilon_{ij} = \{1, -1\}$ such that $\epsilon_{ij} = \epsilon_{ji}$.

Let $p = \mathbb{P}(\epsilon_{ij} = 1)$.

Varying p allows us to *continuously* interpolate between:

Our Ensemble: Signed Toeplitz and Palindromic Toeplitz Matrices

For each entry, assign a randomly chosen $\epsilon_{ij} = \{1, -1\}$ such that $\epsilon_{ij} = \epsilon_{ji}$.

Let $p = \mathbb{P}(\epsilon_{ij} = 1)$.

Varying p allows us to *continuously* interpolate between:

- Real Symmetric at $p = \frac{1}{2}$ (less structured)

Our Ensemble: Signed Toeplitz and Palindromic Toeplitz Matrices

For each entry, assign a randomly chosen $\epsilon_{ij} = \{1, -1\}$ such that $\epsilon_{ij} = \epsilon_{ji}$.

Let $p = \mathbb{P}(\epsilon_{ij} = 1)$.

Varying p allows us to *continuously* interpolate between:

- Real Symmetric at $p = \frac{1}{2}$ (less structured)
- Unsigned Toeplitz/Palindromic Toeplitz at $p = 1$ (more structured)

Our Ensemble: Signed Toeplitz and Palindromic Toeplitz Matrices

For each entry, assign a randomly chosen $\epsilon_{ij} = \{1, -1\}$ such that $\epsilon_{ij} = \epsilon_{ji}$.

Let $p = \mathbb{P}(\epsilon_{ij} = 1)$.

Varying p allows us to *continuously* interpolate between:

- Real Symmetric at $p = \frac{1}{2}$ (less structured)
- Unsigned Toeplitz/Palindromic Toeplitz at $p = 1$ (more structured)

What is the eigenvalue distribution of these signed ensembles?

Markov's Method of Moments

- The k^{th} moment M_k of a probability distribution $f(x)$ defined on an interval $[a, b]$ is $\int_a^b x^k f(x)\,dx$.

Markov's Method of Moments

- The k^{th} moment M_k of a probability distribution $f(x)$ defined on an interval $[a, b]$ is $\int_a^b x^k f(x)\, dx$.
- Show a typical eigenvalue measure $\mu_{A,N}(x)$ converges to a probability distribution P by controlling convergence of average moments of the measures as $N \to \infty$ to the moments of P.

Moments of the Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

Moments of the Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

To each A, we can thus write the eigenvalue distribution as:

$$\mu_{A,N}(x) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right).$$

Moments of the Eigenvalue Distribution

Then the k^{th} moment of $\mu_{A,N}(x)$ is:

Moments of the Eigenvalue Distribution

Then the k^{th} moment of $\mu_{A,N}(x)$ is:

$$M_{N,k}(A) = \int x^k \mu_{A,N}(x)\, dx$$

Moments of the Eigenvalue Distribution

Then the k^{th} moment of $\mu_{A,N}(x)$ is:

$$\begin{aligned} M_{N,k}(A) &= \int x^k \mu_{A,N}(x)\, dx \\ &= \frac{1}{N}\sum_{i=1}^{N}\int_{-\infty}^{\infty} x^k \delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right) dx \end{aligned}$$

Moments of the Eigenvalue Distribution

Then the k^{th} moment of $\mu_{A,N}(x)$ is:

$$\begin{aligned} M_{N,k}(A) &= \int x^k \mu_{A,N}(x)\, dx \\ &= \frac{1}{N}\sum_{i=1}^{N}\int_{-\infty}^{\infty} x^k \delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right) dx \\ &= \frac{1}{N}\sum_{i=1}^{N}\frac{\lambda_i(A)^k}{\left(2\sqrt{N}\right)^k} \end{aligned}$$

Eigenvalue Trace Lemma

For any non-negative integer k, if A is an $N \times N$ matrix with eigenvalues $\lambda_i(A)$, then

$$Trace\left(A^k\right) = \sum_{i=1}^{N} \lambda_i(A)^k .$$

Eigenvalue Trace Lemma

For any non-negative integer k, if A is an $N \times N$ matrix with eigenvalues $\lambda_i(A)$, then

$$Trace\left(A^k\right) = \sum_{i=1}^{N} \lambda_i(A)^k.$$

Thus,

$$M_{N,k}(A) = \frac{\mathrm{Trace}\left(A^k\right)}{2^k N^{\frac{k}{2}+1}} = \frac{\sum_{1 \le i_1,\ldots,i_k \le N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}}{2^k N^{\frac{k}{2}+1}}$$

Eigenvalue Trace Lemma

For any non-negative integer k, if A is an $N \times N$ matrix with eigenvalues $\lambda_i(A)$, then

$$Trace\left(A^k\right) = \sum_{i=1}^{N} \lambda_i(A)^k .$$

Thus,

$$M_{N,k}(A) = \frac{\mathrm{Trace}\left(A^k\right)}{2^k N^{\frac{k}{2}+1}} = \frac{\sum_{1\le i_1,\dots,i_k\le N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}}{2^k N^{\frac{k}{2}+1}}$$

so the average k^{th} moment, $M_k(N) = \mathbb{E}\left[M_{N,k}(A_N)\right]$ is:

Eigenvalue Trace Lemma

For any non-negative integer k, if A is an $N \times N$ matrix with eigenvalues $\lambda_i(A)$, then

$$Trace\left(A^k\right) = \sum_{i=1}^{N} \lambda_i(A)^k .$$

Thus,

$$M_{N,k}(A) = \frac{\text{Trace}\left(A^k\right)}{2^k N^{\frac{k}{2}+1}} = \frac{\sum_{1 \leq i_1,\ldots,i_k \leq N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}}{2^k N^{\frac{k}{2}+1}}$$

so the average k^{th} moment, $M_k(N) = \mathbb{E}\left[M_{N,k}(A_N)\right]$ is:

$$\frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \leq i_1,\ldots,i_k \leq N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1-i_2|} \epsilon_{i_2 i_3} b_{|i_2-i_3|} \ldots \epsilon_{i_k i_1} b_{|i_k-i_1|}\right)$$

Evaluating the Moments

$$M_k(N) = \frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \le i_1, \ldots, i_k \le N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \ldots \epsilon_{i_k i_1} b_{|i_k - i_1|}\right)$$

Evaluating the Moments

$$M_k(N) = \frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \le i_1, \ldots, i_k \le N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \ldots \epsilon_{i_k i_1} b_{|i_k - i_1|}\right)$$

- N^k terms

Evaluating the Moments

$$M_k(N) = \frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \leq i_1,\ldots,i_k \leq N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \ldots \epsilon_{i_k i_1} b_{|i_k - i_1|}\right)$$

- N^k terms
- We look at groups of these terms, "configurations," that all have the same contribution.

Evaluating the Moments

$$M_k(N) = \frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \le i_1,\ldots,i_k \le N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1-i_2|} \epsilon_{i_2 i_3} b_{|i_2-i_3|} \ldots \epsilon_{i_k i_1} b_{|i_k-i_1|}\right)$$

- N^k terms
- We look at groups of these terms, "configurations," that all have the same contribution.
  - What is their contribution?

Evaluating the Moments

$$M_k(N) = \frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \le i_1, \ldots, i_k \le N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \ldots \epsilon_{i_k i_1} b_{|i_k - i_1|}\right)$$

- N^k terms
- We look at groups of these terms, "configurations," that all have the same contribution.
    - What is their contribution?
    - How many terms have this configuration?

Which configurations contribute in the limit?

$$M_k(N) = \frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \le i_1, \dots, i_k \le N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \dots \epsilon_{i_k i_1} b_{|i_k - i_1|}\right)$$

Which configurations contribute in the limit?

$$M_k(N) = \frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \le i_1, \dots, i_k \le N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \cdots \epsilon_{i_k i_1} b_{|i_k - i_1|}\right)$$

What configurations have nonzero contributions?

Which configurations contribute in the limit?

$$M_k(N) = \frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \le i_1, \dots, i_k \le N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \dots \epsilon_{i_k i_1} b_{|i_k - i_1|}\right)$$

What configurations have nonzero contributions?

- The b's must be matched in at least pairs since $\mathbb{E}(b_{ij}) = 0$.

Which configurations contribute in the limit?

$$M_k(N) = \frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \le i_1, \dots, i_k \le N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \dots \epsilon_{i_k i_1} b_{|i_k - i_1|}\right)$$

What configurations have nonzero contributions?

- The b's must be matched in at least pairs since $\mathbb{E}(b_{ij}) = 0$.

What configurations have at least order of magnitude $N^{\frac{k}{2}+1}$ terms?

Which configurations contribute in the limit?

$$M_k(N) = \frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \leq i_1, \dots, i_k \leq N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \cdots \epsilon_{i_k i_1} b_{|i_k - i_1|}\right)$$

What configurations have nonzero contributions?

- The b's must be matched in at least pairs since $\mathbb{E}(b_{ij}) = 0$.

What configurations have at least order of magnitude $N^{\frac{k}{2}+1}$ terms?

- The b's must be matched in at most pairs since there are exactly $\frac{k}{2} + 1$ degrees of freedom when they are matched in exactly pairs.

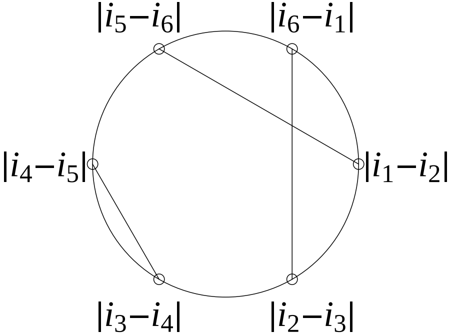
Thus:

Thus:

- Odd moments vanish.

Thus:

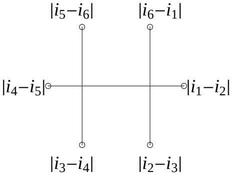
- Odd moments vanish.
- For the even moments M_{2k} we can represent each contributing term as a pairing of $2k$ vertices on a circle as follows:

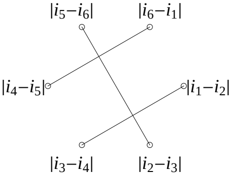


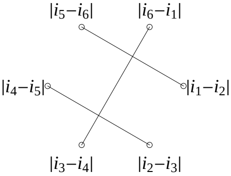
Circle Configurations

Pairings that are the same up to relabelling (**configurations**) have the same contribution:

For example:



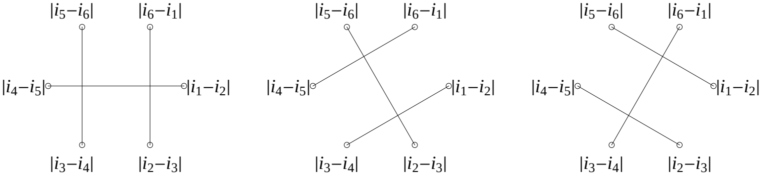




Circle Configurations

Pairings that are the same up to relabelling (**configurations**) have the same contribution:

For example:



Semicircle: Only non-crossing configurations contribute 1

Gaussian: All configurations contribute 1

Weighted Contributions

Theorem:

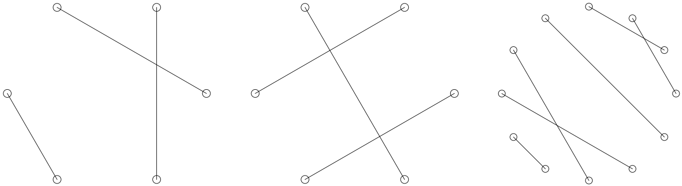
Each configuration weighted by $(2p - 1)^{2m}$, where $2m$ is the number of points on the circle whose edge crosses another edge.

Weighted Contributions

Theorem:

Each configuration weighted by $(2p-1)^{2m}$, where $2m$ is the number of points on the circle whose edge crosses another edge.

Example:



2m = 4 2m=6 2m=8

Proof of Weighted Contributions Theorem

For ϵ_{ij} to be matched with ϵ_{kl} (we know that $\epsilon_{ij} = \epsilon_{kl}$), it must be true that either $i = k$ and $j = l$ or $i = l$ and $j = k$.

Proof of Weighted Contributions Theorem

For ϵ_{ij} to be matched with ϵ_{kl} (we know that $\epsilon_{ij} = \epsilon_{kl}$), it must be true that either $i = k$ and $j = l$ or $i = l$ and $j = k$.

If ϵ_{ij} is matched with some ϵ_{kl}, then $\mathbb{E}\left(\epsilon_{ij}\epsilon_{kl}\right) = 1$.

Proof of Weighted Contributions Theorem

For ϵ_{ij} to be matched with ϵ_{kl} (we know that $\epsilon_{ij} = \epsilon_{kl}$), it must be true that either $i = k$ and $j = l$ or $i = l$ and $j = k$.

If ϵ_{ij} is matched with some ϵ_{kl}, then $\mathbb{E}\left(\epsilon_{ij}\epsilon_{kl}\right) = 1$.

If ϵ_{ij} is not matched with any ϵ_{kl}, then $\mathbb{E}\left(\epsilon_{ij}\right) = (2p - 1)$.

Proof of Weighted Contributions Theorem

For ϵ_{ij} to be matched with ϵ_{kl} (we know that $\epsilon_{ij} = \epsilon_{kl}$), it must be true that either $i = k$ and $j = l$ or $i = l$ and $j = k$.

If ϵ_{ij} is matched with some ϵ_{kl}, then $\mathbb{E}\left(\epsilon_{ij}\epsilon_{kl}\right) = 1$.

If ϵ_{ij} is not matched with any ϵ_{kl}, then $\mathbb{E}\left(\epsilon_{ij}\right) = (2p - 1)$.

Want to prove that two ϵ's are matched if and only if their b's are not in a crossing.

Proof of Weighted Contributions Theorem

A non-crossing pair of b's must have matched ϵs:

Proof of Weighted Contributions Theorem

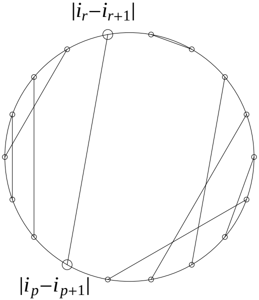
A non-crossing pair of b's must have matched ϵs:

Assume $b_{|i_r - i_{r+1}|}$ and $b_{|i_p - i_{p+1}|}$ are a non-crossing pair.

Proof of Weighted Contributions Theorem

A non-crossing pair of *b*'s must have matched ϵs:

Assume $b_{|i_r - i_{r+1}|}$ and $b_{\left|i_p - i_{p+1}\right|}$ are a non-crossing pair.

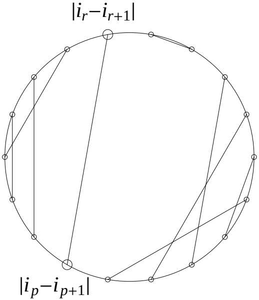


Proof of Weighted Contributions Theorem

A non-crossing pair of *b*'s must have matched ϵs:

Assume $b_{|i_r - i_{r+1}|}$ and $b_{|i_p - i_{p+1}|}$ are a non-crossing pair.

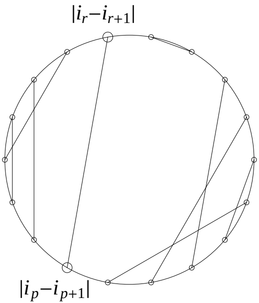
$$\sum_{k=r}^{p} (i_k - i_{k+1}) = 0$$



Proof of Weighted Contributions Theorem

A non-crossing pair of *b*'s must have matched ϵs:

Assume $b_{|i_r - i_{r+1}|}$ and $b_{|i_p - i_{p+1}|}$ are a non-crossing pair.

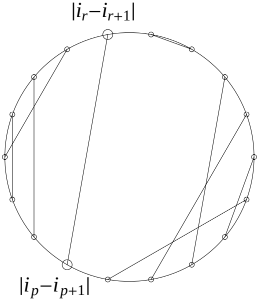


$$\sum_{k=r}^{p} (i_k - i_{k+1}) = 0$$
$$= i_r - i_{r+1} + i_{r+1} \cdots + i_p - i_{p+1} = i_r - i_{p+1}$$

Proof of Weighted Contributions Theorem

A non-crossing pair of b's must have matched ϵs:

Assume $b_{|i_r - i_{r+1}|}$ and $b_{\left|i_p - i_{p+1}\right|}$ are a non-crossing pair.



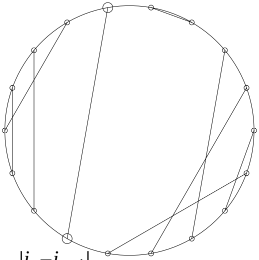
$$\sum_{k=r}^{p} (i_k - i_{k+1}) = 0$$
$$= i_r - i_{r+1} + i_{r+1} \cdots + i_p - i_{p+1} = i_r - i_{p+1}$$

This implies that $i_r = i_{p+1}$.

Proof of Weighted Contributions Theorem

A non-crossing pair of b's must have matched ϵs:

Assume $b_{|i_r - i_{r+1}|}$ and $b_{\left|i_p - i_{p+1}\right|}$ are a non-crossing pair.



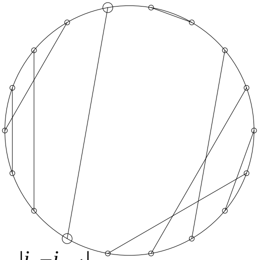
$$\sum_{k=r}^{p} (i_k - i_{k+1}) = 0$$
$$= i_r - i_{r+1} + i_{r+1} \cdots + i_p - i_{p+1} = i_r - i_{p+1}$$

This implies that $i_r = i_{p+1}$.
Similarly, $i_{r+1} = i_p$

Proof of Weighted Contributions Theorem

A non-crossing pair of b's must have matched ϵs:

Assume $b_{|i_r - i_{r+1}|}$ and $b_{|i_p - i_{p+1}|}$ are a non-crossing pair.



$$\sum_{k=r}^{p} (i_k - i_{k+1}) = 0$$
$$= i_r - i_{r+1} + i_{r+1} \cdots + i_p - i_{p+1} = i_r - i_{p+1}$$

This implies that $i_r = i_{p+1}$.
Similarly, $i_{r+1} = i_p$

Thus, $\epsilon_{i_r i_{r+1}} = \epsilon_{i_p i_{p+1}}$.

Proof of Weighted Contributions Theorem

A matched pair of ϵs must not be in a crossing:

Suppose $\epsilon_{i_a i_{a+1}} = \epsilon_{i_b i_{b+1}}$, with $a < b$.

Proof of Weighted Contributions Theorem

A matched pair of ϵs must not be in a crossing:

Suppose $\epsilon_{i_a i_{a+1}} = \epsilon_{i_b i_{b+1}}$, with $a < b$.

$$\begin{aligned}\sum_{k=a}^{b}(i_k - i_{k+1}) &= i_a - i_{b+1} = 0 \\ &= \textstyle\sum_{k=b}^{d} \delta_k |i_k - i_{k+1}|\end{aligned}$$

where $\delta_k = 0$ if and only if the vertex k is paired with is between a and b.

Proof of Weighted Contributions Theorem

A matched pair of ϵs must not be in a crossing:

Suppose $\epsilon_{i_a i_{a+1}} = \epsilon_{i_b i_{b+1}}$, with $a < b$.

$$\sum_{k=a}^{b}(i_k - i_{k+1}) = i_a - i_{b+1} = 0$$

$$= \sum_{k=b}^{d} \delta_k |i_k - i_{k+1}|$$

where $\delta_k = 0$ if and only if the vertex k is paired with is between a and b.

Need N^{k+1} degrees of freedom, so $\delta_k = 0$ for all k.

Proof of Weighted Contributions Theorem

A matched pair of ϵs must not be in a crossing:

Suppose $\epsilon_{i_a i_{a+1}} = \epsilon_{i_b i_{b+1}}$, with $a < b$.

$$\sum_{k=a}^{b} (i_k - i_{k+1}) = i_a - i_{b+1} = 0$$
$$= \sum_{k=b}^{d} \delta_k |i_k - i_{k+1}|$$

where $\delta_k = 0$ if and only if the vertex k is paired with is between a and b.

Need N^{k+1} degrees of freedom, so $\delta_k = 0$ for all k. Thus, $\epsilon_{i_a i_{a+1}}$ and $\epsilon_{i_b i_{b+1}}$ are not in a crossing.

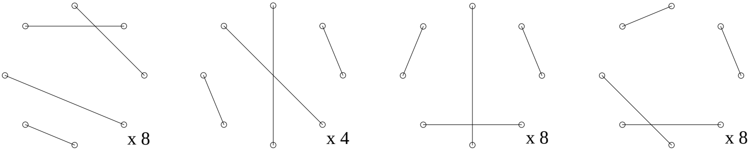
Counting Crossing Configurations

Problem: Out of the $(2k-1)!!$ ways to pair $2k$ vertices, how many will have $2m$ vertices crossing ($Cross_{2k,2m}$)?

Counting Crossing Configurations

Problem: Out of the $(2k-1)!!$ ways to pair $2k$ vertices, how many will have $2m$ vertices crossing ($Cross_{2k,2m}$)?

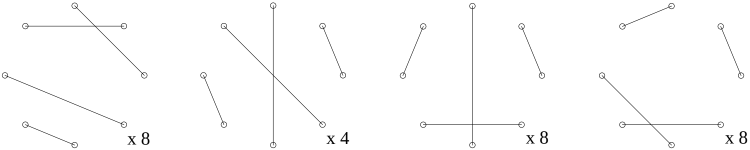
Example: $Cross_{8,4} = 28$



Counting Crossing Configurations

Problem: Out of the $(2k-1)!!$ ways to pair $2k$ vertices, how many will have $2m$ vertices crossing ($Cross_{2k,2m}$)?

Example: $Cross_{8,4} = 28$



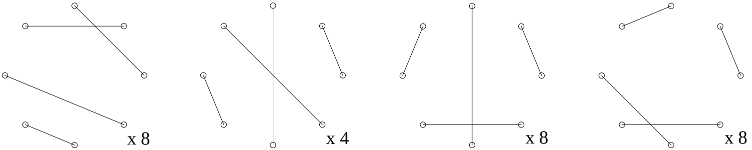
Fact:

$Cross_{2k,0} = C_k$, the k^{th} Catalan number.

Counting Crossing Configurations

Problem: Out of the $(2k-1)!!$ ways to pair $2k$ vertices, how many will have $2m$ vertices crossing ($Cross_{2k,2m}$)?

Example: $Cross_{8,4} = 28$



Fact:

$Cross_{2k,0} = C_k$, the k^{th} Catalan number.

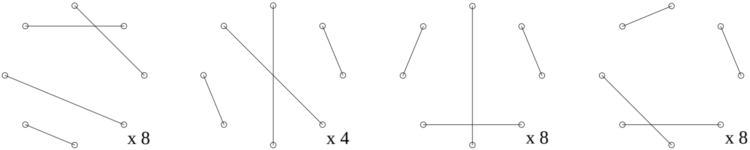
What about for higher m?

Non-Crossing Regions

Theorem:

Suppose $2m$ vertices are already paired in some configuration. The number of ways to pair and place the remaining $2k - 2m$ vertices such that none of them are involved in a crossing is $\binom{2k}{k-m}$.

Example: There are $\binom{8}{2} = 28$ pairings with 4 vertices arranged in a crossing.



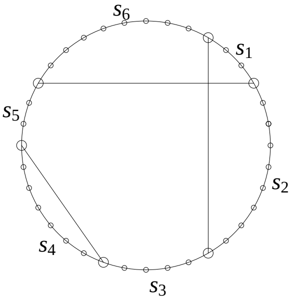
Proof of Non-Crossing Regions Theorem

We showed the following equivalence:

Proof of Non-Crossing Regions Theorem

We showed the following equivalence:

$$\sum_{s_1+s_2+\cdots+s_{2m}=2k-2m} C_{s_1}C_{s_2}\cdots C_{s_{2m}} = \binom{2k}{k-m}.$$



Counting Crossing Configurations

Counting Crossing Configurations

To calculate $Cross_{2k,2m}$, we write it as the following sum:

$$Cross_{2k,2m} = \sum_{p=1}^{\lfloor \frac{m}{4} \rfloor} P_{2k,2m,p}.$$

where $P_{2k,2m,p}$ is the number of configurations of $2k$ vertices with $2m$ vertices crossing in p partitions.

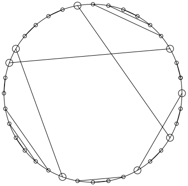
Counting Crossing Configurations

To calculate $Cross_{2k,2m}$, we write it as the following sum:

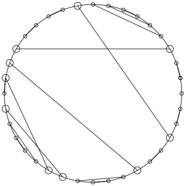
$$Cross_{2k,2m} = \sum_{p=1}^{\lfloor \frac{m}{4} \rfloor} P_{2k,2m,p}.$$

where $P_{2k,2m,p}$ is the number of configurations of $2k$ vertices with $2m$ vertices crossing in p partitions.

For example:



p = 1

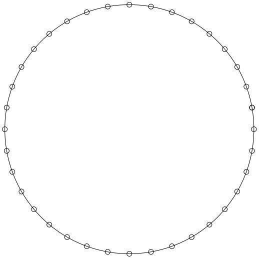


p = 2

We then apply our theorem to get formulas for $P_{2k,2m,p}$.
For example:

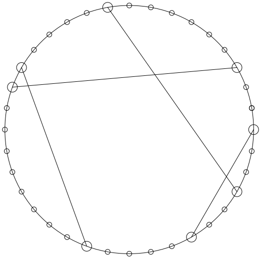
We then apply our theorem to get formulas for $P_{2k,2m,p}$.
For example:

$$P_{2k,2m,1} = \qquad\qquad .$$



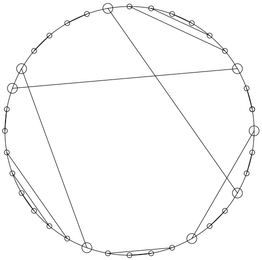
We then apply our theorem to get formulas for $P_{2k,2m,p}$.
For example:

$$P_{2k,2m,1} = Cross_{2m,2m} \qquad .$$



We then apply our theorem to get formulas for $P_{2k,2m,p}$. For example:

$$P_{2k,2m,1} = Cross_{2m,2m}\binom{2k}{k-m}.$$



Counting Crossing Configurations

For:

$2k \setminus 2m$	0	4	6	8	10	Total
2						1
4						3
6						15
8						105
10						945
$\vdots$						

Counting Crossing Configurations

For:

$2k \setminus 2m$	0	4	6	8	10	Total
2	1					1
4	2					3
6	5					15
8	14					105
10	42					945
$\vdots$						

Counting Crossing Configurations

For:

$2k \setminus 2m$	0	4	6	8	10	Total
2	1					1
4	2	1				3
6	5					15
8	14					105
10	42					945
$\vdots$						

Counting Crossing Configurations

For:

- $2m = 4$, there are $\binom{2k}{k-2}$ such pairings.

$2k \setminus 2m$	0	4	6	8	10	Total
2	1					1
4	2	1				3
6	5					15
8	14					105
10	42					945
$\vdots$						

Counting Crossing Configurations

For:

- $2m = 4$, there are $\binom{2k}{k-2}$ such pairings.

$2k \setminus 2m$	0	4	6	8	10	Total
2	1					1
4	2	1				3
6	5	6				15
8	14	28				105
10	42	120				945
$\vdots$						

Counting Crossing Configurations

For:

- $2m = 4$, there are $\binom{2k}{k-2}$ such pairings.

$2k \setminus 2m$	0	4	6	8	10	Total
2	1					1
4	2	1				3
6	5	6	4			15
8	14	28				105
10	42	120				945
⋮						

Counting Crossing Configurations

For:

- $2m = 4$, there are $\binom{2k}{k-2}$ such pairings.
- $2m = 6$, there are $4\binom{2k}{k-3}$ such pairings.

$2k \setminus 2m$	0	4	6	8	10	Total
2	1					1
4	2	1				3
6	5	6	4			15
8	14	28				105
10	42	120				945
$\vdots$						

Counting Crossing Configurations

For:

- $2m = 4$, there are $\binom{2k}{k-2}$ such pairings.
- $2m = 6$, there are $4\binom{2k}{k-3}$ such pairings.

$2k \setminus 2m$	0	4	6	8	10	Total
2	1					1
4	2	1				3
6	5	6	4			15
8	14	28	32			105
10	42	120	180			945
$\vdots$						

Counting Crossing Configurations

For:

- $2m = 4$, there are $\binom{2k}{k-2}$ such pairings.
- $2m = 6$, there are $4\binom{2k}{k-3}$ such pairings.

$2k \setminus 2m$	0	4	6	8	10	Total
2	1					1
4	2	1				3
6	5	6	4			15
8	14	28	32	31		105
10	42	120	180			945
⋮						

Counting Crossing Configurations

For:

- $2m = 4$, there are $\binom{2k}{k-2}$ such pairings.
- $2m = 6$, there are $4\binom{2k}{k-3}$ such pairings.
- $2m = 8$, there are $31\binom{2k}{k-4} + \frac{1}{2}\sum_{i=0}^{k-5}\binom{2k}{i}(2k-2i)$

$2k \setminus 2m$	0	4	6	8	10	Total
2	1					1
4	2	1				3
6	5	6	4			15
8	14	28	32	31		105
10	42	120	180			945
⋮						

Counting Crossing Configurations

For:

- $2m = 4$, there are $\binom{2k}{k-2}$ such pairings.
- $2m = 6$, there are $4\binom{2k}{k-3}$ such pairings.
- $2m = 8$, there are $31\binom{2k}{k-4} + \frac{1}{2}\sum_{i=0}^{k-5}\binom{2k}{i}(2k-2i)$

$2k \setminus 2m$	0	4	6	8	10	Total
2	1					1
4	2	1				3
6	5	6	4			15
8	14	28	32	31		105
10	42	120	180	315		945
$\vdots$						

Counting Crossing Configurations

For:

- $2m = 4$, there are $\binom{2k}{k-2}$ such pairings.
- $2m = 6$, there are $4\binom{2k}{k-3}$ such pairings.
- $2m = 8$, there are $31\binom{2k}{k-4} + \frac{1}{2}\sum_{i=0}^{k-5}\binom{2k}{i}(2k-2i)$

$2k \setminus 2m$	0	4	6	8	10	Total
2	1					1
4	2	1				3
6	5	6	4			15
8	14	28	32	31		105
10	42	120	180	315	288	945
$\vdots$						

Counting Crossing Configurations

For:

- $2m = 4$, there are $\binom{2k}{k-2}$ such pairings.
- $2m = 6$, there are $4\binom{2k}{k-3}$ such pairings.
- $2m = 8$, there are $31\binom{2k}{k-4} + \frac{1}{2}\sum_{i=0}^{k-5}\binom{2k}{i}(2k-2i)$
- $2m = 10$, there are $288\binom{2k}{k-5} + 4\sum_{i=0}^{k-6}\binom{2k}{i}(2k-2i)$

$2k \setminus 2m$	0	4	6	8	10	Total
2	1					1
4	2	1				3
6	5	6	4			15
8	14	28	32	31		105
10	42	120	180	315	288	945
⋮						

Counting Crossing Configurations

For:

- $2m = 4$, there are $\binom{2k}{k-2}$ such pairings.
- $2m = 6$, there are $4\binom{2k}{k-3}$ such pairings.
- $2m = 8$, there are $31\binom{2k}{k-4} + \frac{1}{2}\sum_{i=0}^{k-5}\binom{2k}{i}(2k-2i)$
- $2m = 10$, there are $288\binom{2k}{k-5} + 4\sum_{i=0}^{k-6}\binom{2k}{i}(2k-2i)$

$2k \setminus 2m$	0	4	6	8	10	Total
2	1					1
4	2	1				3
6	5	6	4			15
8	14	28	32	31		105
10	42	120	180	315	288	945
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Summary of Results

Summary of Results

- $p = \frac{1}{2}$: Semicircle Distribution (Bounded Support)
  $p \neq \frac{1}{2}$: Unbounded Support

Summary of Results

- $p = \frac{1}{2}$: Semicircle Distribution (Bounded Support)
  $p \neq \frac{1}{2}$: Unbounded Support
- A method to calculate the moments of the eigenvalue distribution, from which we can recover the distribution

Summary of Results

- $p = \frac{1}{2}$: Semicircle Distribution (Bounded Support)
  $p \neq \frac{1}{2}$: Unbounded Support
- A method to calculate the moments of the eigenvalue distribution, from which we can recover the distribution
  - Weight of each configuration as a function of p and the number of vertices in a crossing ($2m$): $(2p-1)^{2m}$

Summary of Results

- $p = \frac{1}{2}$: Semicircle Distribution (Bounded Support)
  $p \neq \frac{1}{2}$: Unbounded Support
- A method to calculate the moments of the eigenvalue distribution, from which we can recover the distribution
  - Weight of each configuration as a function of p and the number of vertices in a crossing ($2m$): $(2p-1)^{2m}$
  - A way to count the number of configurations with $2m$ vertices crossing for all m

Summary of Results

- $p = \frac{1}{2}$: Semicircle Distribution (Bounded Support)
  $p \neq \frac{1}{2}$: Unbounded Support
- A method to calculate the moments of the eigenvalue distribution, from which we can recover the distribution
  - Weight of each configuration as a function of p and the number of vertices in a crossing ($2m$): $(2p - 1)^{2m}$
  - A way to count the number of configurations with $2m$ vertices crossing for all m
- Limiting behavior of the mean and variance of the moments, giving bounds for the moments

Many thanks to:

- YMC, Ohio State University
- Williams College, SMALL 2011
- National Science Foundation
- Professor Steven J Miller