Generalized Ramanujan Primes

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- We expect linearly increasing intervals to contain an increasing amount of primes.

Historical Introduction

Bertrand's Postulate (1845)

For all integers $x \geq 2$, there exists at least one prime in $(x/2, x]$.

Ramanujan Primes

Definition

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- Ramanujan: For each integer n, R_n exists.
- Sondow: $R_n \sim p_{2n}$.
- Sondow: As $n \to \infty$, 50% of primes are Ramanujan.

c-Ramanujan Primes

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  $R_{c,n} \sim p_{\frac{n}{1-c}}$

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  Given $R_{c,n} \sim p_{\frac{n}{1-c}}$, it follows $R_{c,N(1-c)} \sim p_N$.

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  1-c

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The Prime Number Theorem states:

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} = 1.$$

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The Prime Number Theorem gives us

$$\pi(x) = \mathrm{Li}(x) \; + \; O\left(\frac{x}{\log^2 x}\right),$$

i.e., there is a $C > 0$ such that for all x sufficiently large

$$-C\frac{x}{\log x} \;\leq\; \pi(x) - \mathrm{Li}(x) \;\leq\; C\frac{x}{\log x}.$$

Existence of $R_{c,n}$

Theorem (Amersi, Beckwith, Ronan, 2011)

For all $n \in \mathbb{Z}$ and all $c \in (0, 1)$, the n-th c-Ramanujan prime $R_{c,n}$ exists.

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- The number of primes in $(cx, x]$ is $\pi(x) - \pi(cx)$.
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- Using the Prime Number Theorem and Mean Value Theorem:

$$\begin{aligned} \pi(x) - \pi(cx) &= \mathrm{Li}(x) - \mathrm{Li}(cx) + O(x \log^{-2} x) \\ &= \mathrm{Li}'(y_c)(x - cx) + O(x \log^{-2} x) \\ &\quad \text{with } y_c(x) \in [cx, x]. \end{aligned}$$

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- For sufficiently large x, $\pi(x) - \pi(cx)$ is strictly increasing and $\pi(x) - \pi(cx) \geq n$, for all integers n.

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- We then obtain bounds on $\log R_{c,n}$:

$$\left(1 - \frac{\beta_c \log\log n}{\log n}\right) \log n \leq \log R_{c,n} \leq \left(1 + \frac{\beta_c \log\log n}{\log n}\right) \log n.$$

Asymptotic Behavior

Theorem (Amersi, Beckwith, Ronan, 2011)

For any fixed $c \in (0, 1)$, the nth c-Ramanujan prime is asymptotic to the $\frac{n}{1-c}$th prime as $n \to \infty$

By the triangle inequality

$$\begin{aligned} \left|R_{c,n} - p_{\frac{n}{1-c}}\right| \quad \leq \quad & \left|R_{c,n} - \frac{n}{1-c}\log R_{c,n}\right| + \left|\frac{n}{1-c}\log R_{c,n} - \frac{n}{1-c}\log n\right| \\ & + \left|\frac{n}{1-c}\log n - \frac{n}{1-c}\log \frac{n}{1-c}\right| \\ & + \left|\frac{n}{1-c}\log n - p_{\frac{n}{1-c}}\right| \end{aligned}$$

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Since $\frac{n\log\log n}{p_n} \to 0$ as $n \to \infty \Rightarrow R_{c,n} \sim p_{\frac{n}{1-c}}$

Frequency of c-Ramanujan Primes

Theorem (Amersi, Beckwith, Ronan, 2011)

In the limit, the probability of a generic prime being a c-Ramanujan prime is $1 - c$.

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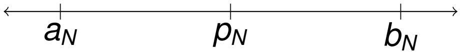

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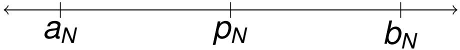

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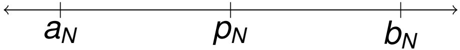

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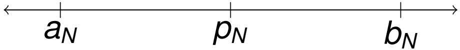

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- Goal: $\frac{\pi(b_N)-\pi(a_N)}{\pi(p_N)} \to 0$ as $N \to \infty$.

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$$\Rightarrow \frac{\pi(b_N) - \pi(a_N)}{\pi(p_N)} \leq \xi \frac{\log \log N}{\log N} \to 0 \text{ as } N \to \infty$$

Prime Numbers

2	3	5	7	11	13	17
19	23	29	31	37	41	43
47	53	59	61	67	71	73
79	83	89	97	101	103	107
109	113	127	131	137	139	149
151	157	163	167	173	179	181
191	193	197	199	211	223	227

Ramanujan Primes

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$$\mathbb{E}[L_N] \quad \approx \quad \frac{\log N}{\log(1/P)} - \left(\frac{1}{2} - \frac{\log(1-P)+\gamma}{\log(1/P)}\right)$$

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$$\mathrm{Var}[L_N] \approx \frac{\pi^2}{6\log^2(1/P)} + \frac{1}{12}$$

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- Define P_c as the frequency of c-Ramanujan primes amongst the primes,
- As $N \to \infty, P_c = 1 - c$,
- For finite intervals $[a, b]$, P_c is a function of a and b,
- Choose $a = 10^5, b = 10^6$.

Distribution of generalized Ramanujan primes

	Length of the longest run in $[10^5, 10^6]$ of			
	c-Ramanujan primes		Non-c-Ramanujan primes	
c	Expected	Actual	Expected	Actual
0.50	14	20	16	36

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c	Expected	Actual	Expected	Actual
0.10	70	58	5	3
0.20	38	36	7	7
0.30	25	25	10	12
0.40	18	21	13	16
0.50	14	20	16	36
0.60	11	17	22	42
0.70	9	14	30	78
0.80	7	9	46	154
0.90	5	11	91	345

Open Problems

1. Laishram and Sondow: $p_{2n} < R_n < p_{3n}$ for $n > 1$. Can we find good choices of a_c and b_c such that $p_{a_c n} \leq R_{c,n} \leq p_{b_c n}$ for all n?

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2. For a given prime p, for what values of c is p a c-Ramanujan prime?
3. Is there any explanation for the unexpected distribution of c-Ramanujan primes amongst the primes?

Acknowledgements

This work was supported by the NSF, Williams College, University College London.

We would like to thank Jonathan Sondow and our colleagues from the 2011 REU at Williams College.