Mind the Gap: Distribution of Gaps in Generalized Zeckendorf Decompositions

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Joint with: Olivia Beckwith, Louis Gaudet, Shiyu Li, Steven J. Miller, and Philip Tosteson
http://www.williams.edu/Mathematics/sjmiller/public_html

YMC, Ohio State University, July 27-29th 2012

Introduction

Goals of the Talk

- Overview of recurrences, decompositions and gaps
- Hopping and Kangaroo recurrences
- Interesting probability distributions
- Other positive linear recurrences

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
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Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Previous Generalizations

Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}, \; n \geq L$$

with $H_1 = 1$, $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1, \; n < L$, coefficients $c_i \geq 0$; $c_1, c_L > 0$ if $L \geq 2$; $c_1 > 1$ if $L = 1$.

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- Central Limit Type Theorem

Gaps Between Summands

Distribution of Gaps

For $H_{i_1} + H_{i_2} + \cdots + H_{i_n}$, the gaps are the differences:

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Big Question: What is $P(m) = \lim_{n\to\infty} P_n(m)$?

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- Random Matrix Theory, Physics, and Riemann Zeta Function
- Wait times: banks, lines, managing computer queues

Previous Results (Beckwith-Miller 2011)

Theorem (Zeckendorf Gap Distribution)

For Zeckendorf decompositions, $P(m) = 1/\varphi^m$ for $m \geq 2$, with $\varphi = \frac{1+\sqrt{5}}{2}$ the golden mean.

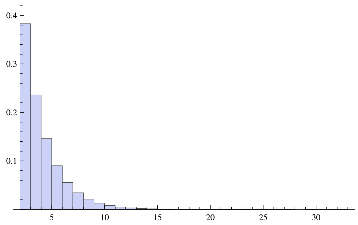

Figure: Distribution of gaps in $[F_{1000}, F_{1001})$.

Kangaroo Recurrences

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$$100 \ = \ K_{10} \ + \ K_7 \ + \ K_4 \ + \ K_2.$$

Only a few kangaroos were harmed in the making of this presentation.

Our Results

Lemma

Given any Kangaroo decomposition for $n \in \mathbb{N}$, $P_n(j) = 0$ for $j < g$.

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We are interested in studying $P_n(j)$ for $j \geq g$ as $n \to \infty$.

Probability of Obtaining a Gap Length $j \geq g+1$

Generalized Binet's Formula: It is well known that we can write

$$K_n \;=\; a_1\lambda_1^n + a_2\lambda_2^n + \cdots + a_{\ell g+1}\lambda_{\ell g+1}^n$$

where $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{\ell g+1}|$.

Let $\lambda_{g,\ell} = \lambda_1$ for a Kangaroo recurrence with ℓ hops of length g.

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Theorem (Exponential Decay)

If $j \geq g+1$, then $P(j) = (\lambda_{g,\ell} - 1)^2 \left(\frac{a_1}{C_{Lek}}\right) \lambda_{g,\ell}^{-j}$.

Proof Set Up

Theorem

If $j \geq g + 1$, *then* $P(j) = (\lambda_{g,\ell} - 1)^2 \left(\frac{a_1}{C_{Lek}}\right) \lambda_{g,\ell}^{-j}$.

Let $X_{i,i+j}(n) = \#\{m \in [K_n, K_{n+1})$: decomposition of m includes K_i, K_{i+j}, but not K_q for $i < q < i + j\}$.

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$$P(j) = \lim_{n\to\infty} \frac{1}{Y(n)} \sum_{i=1}^{n-j} X_{i,i+j}(n).$$

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Generalized Lekkerkerker $\Rightarrow$ $Y(n) \sim (C_{Lek} n + d)(K_{n+1} - K_n)$.

A Quick Counting Lesson: How do we count $X_{i,i+j}$?

We need to see the number of legal decompositions with a gap of length j.

Can count how many legal decompositions exist to the left and right of the gap.

Lemma

Let $H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L}$ be a Positive Linear Recurrence Sequence, then the number of legal decompositions which contain H_m as the largest summand is $H_{m+1} - H_m$.

Calculating $X_{i,i+j}$

Theorem

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How many decompositions contain a gap from K_i to K_{i+j}?

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Right: For the indices greater than $i+j$:
$K_{n-j-i-g+1} - K_{n-j-i-g} + \cdots + K_{n-j-i-\ell g+1} - K_{n-j-i-\ell g}$ choices.

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So $X_{i,i+j}(n) = \textit{Left} * \textit{Right} =$
$(K_{i+1} - K_i)(K_{n-i-j+2} - K_{n-i-j+1} - (K_{n-i-j+1} - K_{n-i-j}))$.

Final Steps of the Proof

For sufficiently large n, g, and ℓ, $K_n \approx a_1 \lambda_{g,\ell}^n$.

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The proof is combinatorial in nature like the previous one.

The main difference is the probabilities of the left and the right are no longer independent.

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$$P(g) = \lim_{n\to\infty} \frac{1}{Y(n)} \sum_{b=1}^{\ell-1} \sum_{i=1}^{n-bg} X_{i,i+bg}(n)$$

where $X_{i,i+bg} = (K_{i-g} - K_1)(K_{n-i-(b+1)g+1} - K_{n-i-(b+1)g})$

Approximating $\lambda_{g,\ell}$

Characteristic polynomial of the recurrence $\Rightarrow$ transcendental equation

$$\lambda_{g,\ell}^{g} \approx \left(1 + \frac{\alpha}{g}\right)^{g},$$

where $\alpha \approx \log(g) - \log(\log(g)) + \frac{\log(\log(g))}{\log(g)}$.

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where $\alpha \approx \log(g) - \log(\log(g)) + \frac{\log(\log(g))}{\log(g)}$.

This tells us that

- $\lambda_{g,\ell} \approx 1$
- $\lambda_{g,\ell}^{-g} \approx \frac{1}{g}$

Approximating $\lambda_{g,\ell}$

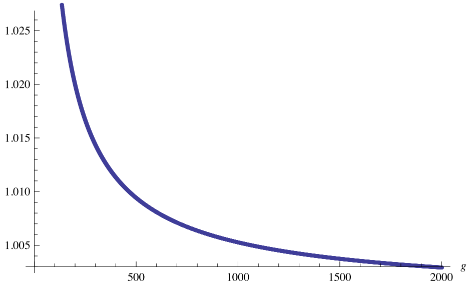

Figure: g vs $\lambda_{g,\ell}$ for large ℓ.

Probability Ratios

What is the probability of getting a gap of length g compared to any other gap?

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$$\frac{\text{Prob(Gap at least } g)}{\text{Prob(Gap at least } g+1)} = \frac{\lambda_{g,\ell}^{-2g}}{\lambda_{g,\ell}^{-g}(\lambda_{g,\ell}-1)(1-\frac{1}{\lambda_{g,\ell}^{n-g-1}})}.$$

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$$\frac{\text{Prob}(\text{Gap at least } g)}{\text{Prob}(\text{Gap at least } g+1)} \approx \frac{1}{\alpha} \approx \frac{1}{\log(g) - \log(\log(g)) + \frac{\log(\log(g))}{\log(g)}}.$$

Other Positive Linear Recurrences

Positive Linear Recurrences of Any Length

Theorem

Let $H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L}$ be a Positive Linear Recurrence Sequence, then, if $j \geq L$, $P(j) = (\lambda_1 - 1)^2 \left(\frac{a_1}{C_{Lek}}\right) \lambda_1^{-j}$, where λ_1 is the largest root of the characteristic polynomial of the recurrence.

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What can we say about the distribution of gaps $< L$ for any PLRS?

Positive Linear Recurrences of Any Length

Theorem

Let $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n+1-L}$ be a positive linear recurrence of length L where $c_i \geq 1$ for all $1 \leq i \leq L$. Then

$$P(j) = \begin{cases} 1 - (\frac{a_1}{C_{Lek}})(\lambda_1^{-n+2} - \lambda_1^{-n+1} + 2\lambda_1^{-1} + a_1^{-1} - 3) & \text{for } j = 0 \\ \lambda_1^{-1}(\frac{1}{C_{Lek}})(\lambda_1(1 - 2a_1) + a_1) & \text{for } j = 1 \\ (\lambda_1 - 1)^2 \left(\frac{a_1}{C_{Lek}}\right) \lambda_1^{-j} & \text{for } j \geq 2 \end{cases}$$

Positive Linear Recurrences of Length 2

We can calculate the constants $\lambda_1, \lambda_2, a_1$, and C_{Lek} for recurrences of length 2

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$$\lambda_1 = \frac{c_1 + \sqrt{c_1^2 + 4c_2}}{2}$$

$$\lambda_2 = \frac{c_1 - \sqrt{c_1^2 + 4c_2}}{2}$$

$$a_1 = \frac{c_1 + 1 - \lambda_2}{\lambda_1^2 - \lambda_1 \lambda_2}$$

$$C_{Lek} = \frac{((c_1^2 - c_1)\lambda_1) + (2c_1 c_2 + c_2^2 - c_2)}{2c_1\lambda_1 + 4c_2}$$

Future Research

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- Given a specific $m \in \mathbb{N}$, what is the probability its decomposition has gap distribution close to the average?
- What is the average longest gap?
- How do the coefficients in a recurrence affect the results?
- Generalizing results to all PLRS and signed decompositions

Acknowledgements

Thanks to...

- NSF Grant DMS0850577
- NSF Grant DMS0970067
- Young Mathematicians Conference
- Our peers at Williams SMALL REU Summers 2010, 2011, and 2012
- Our Advisor Steven J. Miller

Thanks for your time!